ES.1803 Quiz 7, Spring 2024 Solutions

Unit 1: Problems 1-4. Required

Problem 1. (15 points)

(a) [10] *Find the general solution to* $x'' - 100x = 5$.

Solution: Partictular solution: Try $x(t) = c$ (constant), $\Rightarrow x_p(t) = -5/100$.

Homogeneous solution: characteristic roots $r = \pm 10$, so $x_h(t) = c_1 e^{10t} + c_2 e^{-10t}$.

General solution: $\boxed{x(t) = x_p(t) + x_h(t) = -\frac{5}{100} + c_1 e^{10t} + c_2 e^{-10t}.}$.

(b) [5] *A certain constant coefficient operator* $P(D)$ *is such that all solutions to* $P(D)x = 0$ *are linear combinations of functions of the form* e^{-t} *times sinusoids of angular frequencies 1 and 2.*

What is the characteristic polynomial $P(r)$*?*

For Part (b), you should assume that its leading coefficient is 1. You may leave your answer as a product, but you must eliminate any complex numbers that appear.

Solution: Characteristic roots are $-1 \pm i$, $-1 \pm 2i$.

So, $P(r) = (r + 1 - i)(r + 1 + i)(r + 1 - 2i)(r + 1 + 2i) = \boxed{\left((r + 1)^2 + 1\right)\left((r + 1)^2 + 4\right)}$.

Also okay: $(r + 1)^4 + 5(r + 1)^2 + 4$ or $r^4 + 4r^3 + 11r^2 + 14r + 10$.

Problem 2. (40 points)

(a) [10] *Find the sinusoidal solution of* $\frac{d^3x}{dt^3} + 2\frac{dx}{dt} + x = 5\cos(3t)$

Express your answer in the form $A\cos(\omega t - \phi)$.

Solution: $x_p(t) = \frac{5\cos(3t - \phi)}{|P(3i)|}$, where $\phi = \text{Arg}(P(3i))$.

$P(r) = r^3 + 2r + 1$, so $P(3i) = -27i + 6i + 1 = 1 - 21i$.

So, $|P(3i)| = \sqrt{1 + 21^2} = \sqrt{442}$, $\boxed{\phi = \text{Arg}(1 - 21i) = \tan^{-1}(-21) \text{ in Q4}}$, $\boxed{x_p(t) = \frac{5\cos(3t - \phi)}{\sqrt{442}}}$.

(b) [10] *Find a real-valued solution to* $x'' + x = \cos(t)$.

Solution: $P(r) = r^2 + 1$. So, $P(i) = 0$.

We'll need $P'(r) = 2r$, $P'(i) = 2i$.

Extended SRF: $x_p(t) = \frac{t\cos(t - \phi)}{|P'(i)|}$, where $\phi = \text{Arg}(P'(i))$.

$|P'(i)| = 2$, $\phi = \text{Arg}(P'(i) = \pi/2$.

So, $\boxed{x_p(t) = \frac{t\cos(t - \pi/2)}{2}}$ or $x_p(t) = \frac{t\sin(t)}{2}$.

(c) [10] *Find a real-valued solution to* $x'' + 2x' + 5x = e^t\cos(2t)$.

Solution: Complexify: $z'' + 2z' + 5z = e^{(1+2i)t}, \quad x = \text{Re}(z)$.
$P(1+2i) = (1+2i)^2 + 2(1+2i) + 5 = 4 + 8i$. So,

$$\boxed{|P(1+2i)| = 4\sqrt{5}}, \quad \boxed{\phi = \text{Arg}(P(1+2i)) = \tan^{-1}(2), \text{ in Q1}}.$$

$$\boxed{x_p(t) = \frac{e^t\cos(2t-\phi)}{|P(1+2i)|} = \frac{e^t\cos(2t-\phi)}{4\sqrt{5}}}.$$

(d) [10] *All we know about a certain linear time invariant operator $P(D)$ is that*

$$P(D)\sin(3t) = \cos(3t).$$

Find a solution to $P(D)x = 4\cos(3t - \frac{\pi}{5})$.

Solution: Linear time invariance: $\boxed{x(t) = 4\sin(3t - \pi/5)}$.

Problem 3. (10 points) In this problem m and k are positive constants.

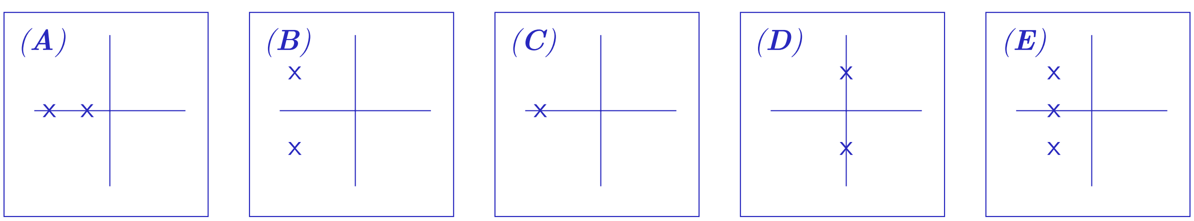
(a) [5] *What type of physical system is modeled by the DE $x' + k\,x = 0$?*
(We're looking for a short answer to this.)

Solution: Exponential decay.

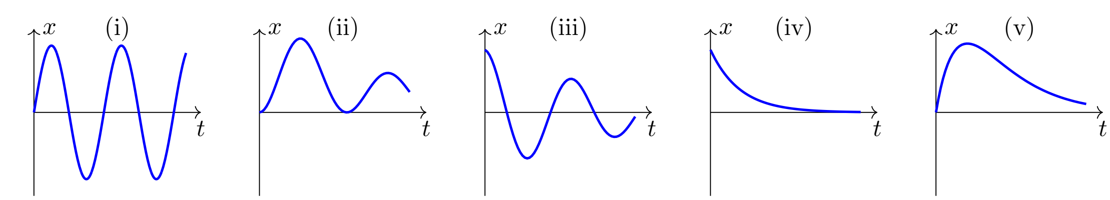
(b) [5] *What type of physical system is modeled by the DE $m\ddot{x} + k\,x = 0$?*
(We're looking for a short answer to this.)

Solution: Simple harmonic oscillator.

Problem 4. (16 points)
Each of the following plots are in the complex plane and give the pole diagram for a linear time invariant system of the form $P(D)x = f$



(a) [10] *Below are graphs of solutions to $P(D)x = 0$ for each of the five systems. Using the labels A-E, label each graph with the letter of the corresponding pole diagram.*



Solution: (i) D. System D is second-order and undamped, so the solutions don't decay. (i) is the only sinusoidal graph.

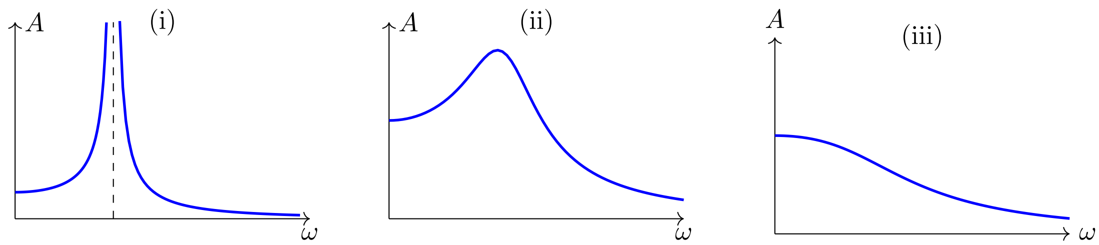
(ii) E. System E is third-order. (ii) is the only graph that doesn't match a first or second-order system.

(iii) B. System D is second-order and underdamped. (iii) is a decaying oscillation.

(iv) C. System C is first-order, exponential decay. (iv) is only such graph.

(v) A. System A is second-order and overdamped. (v) starts at 0 with positive velocity, then it comes to rest and decays towards 0 without crossing the axis. This is the behavior of an overdamped oscillator.

(b) [6] *Below are graphs of the amplitude responses of the systems A, B and D to inputs of the form $f(t) = \cos(\omega t)$. Using the labels match the amplitude response to the correct pole diagram.*



Solution: (i) D. Gain graph (i) shows pure resonance. This implies a pole on the imaginary axis. D is the only system that matches this.

(ii) B. Gain graph (ii) shows practical resonance. For a second-order system this means well underdamped. Only system B has complex roots off the imaginary axis.

(iii) A. System A is overdamped, so the gain graph has no practical resonance. This matches graph (iii).

Unit 5: Problems 5-8. Required

Problem 5. (25 points)
The DE system $x' = xy - 18, \quad y' = x - 2y$ *has critical points (-6,-3) and (6,3).*

(a) [10] *Compute the linearized system at each of the critical points, solve for the eigenvalues and give the type of linearized critical point. Solve for the eigenvectors only if they will be needed in order to get a good sketch of the trajectories in Part (c).*

Solution: The Jacobian $J(x, y) = \begin{bmatrix} y & x \\ 1 & -2 \end{bmatrix}$.

At (-6,-3): $J(-6, -3) = \begin{bmatrix} -3 & -6 \\ 1 & -2 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 5\lambda + 12 = 0$. This has roots: $\lambda = \dfrac{-5 \pm \sqrt{-23}}{2}$.

So we have a $\boxed{\text{linearized spiral sink}}$.

The 1 in the lower left of $J(-6,-3)$ tells us the spiral turns $\boxed{\text{counterclockwise}}$.

At (6,3): $J(6,3) = \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix}$.

Characteristic equation: $\lambda^2 - \lambda - 12 = 0$. This has roots: $\lambda = 4, -3$.

So we have a $\boxed{\text{linearized saddle}}$. For a saddle, we'll need basic eigenvectors, i.e., bases of $J - \lambda I$:

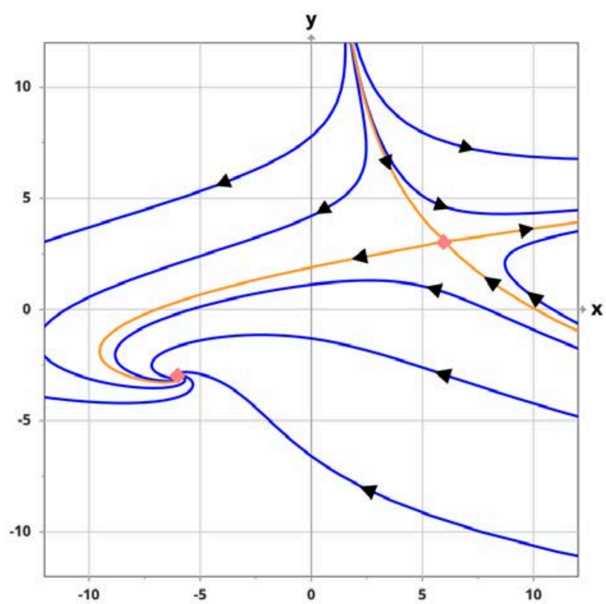
$\lambda = 4$: $\quad J - \lambda I = \begin{bmatrix} -1 & 6 \\ 1 & -6 \end{bmatrix}$. Basic eigenvector: $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$.

$\lambda = -3$: $\quad J - \lambda I = \begin{bmatrix} 6 & 6 \\ 1 & 1 \end{bmatrix}$. Basic eigenvector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) [5] *Will the behavior of the trajectories of the non-linear system near the critical points be essentially the same as that of the linearized system in each case? What property of the linearized system at the critical point allows you to be able to tell in each case?*

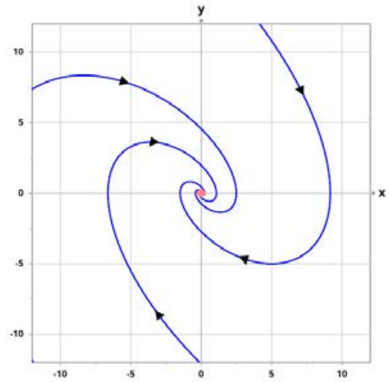
Solution: $\boxed{\text{Yes.}}$ Both linearized systems are structurally stable, so they match the nonlinear system.

(c) *[10] Using the information about the linearized system found in Parts (a) and (b), sketch in (on the axes below) some trajectories in the neighborhood of each critical point. Then use this to create a conjectural picture of the trajectories of the non-linear system.*



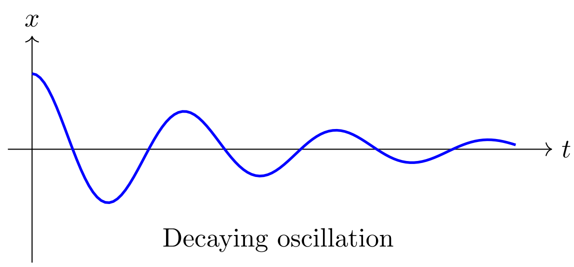
Problem 6. (15 points)

The phase plane portrait shown represents some trajectories for a DE system $\mathbf{x}' = A\mathbf{x}$. *Here* $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$, *a and b are constants, and* $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.



(a) [10] *Using the given phase plane picture, sketch the graph of $x(t)$ corresponding to the solution to the system with IC $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.*

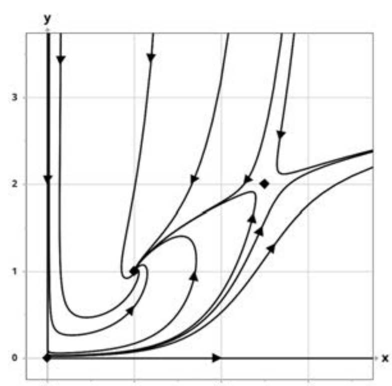
(That is, sketch x vs. t, where $x(t)$ is the first component of a solution $\mathbf{x}(t)$).



(b) [5] *Give a formula expressing y in terms of x.*

Solution: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow x' = y$ or $\boxed{y = x'}$.

Problem 7. (10 points)
The phase portrait shown is for a non-linear system relating the populations of two interacting species. Describe the possible long-term behavior of the two populations.



Solution: Critical points: $(1, 1)$ is spiral sink and $(2.5, 2)$ is a saddle.

Either the populations stabilize at $x = 1$, $y = 1$ or both x and y go to infinity.

More completely:

If x and y are both 0, that's where they stay.

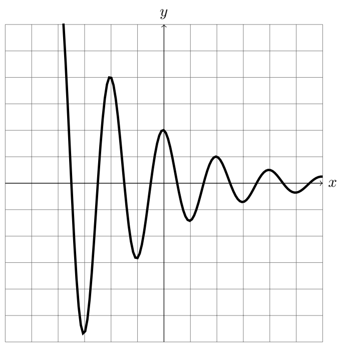
If $x = 0$ and $y > 0$, then x stays at 0 and y decays to 0, i.e., y dies off.

If $x > 0$ and $y = 0$, then x grows to infinity and y stays at 0.

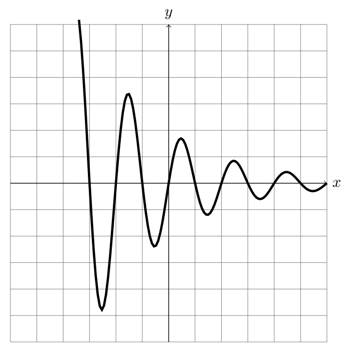
If both populations are positive, then either they stabilize at $x = 1$, $y = 1$ or both x and y go to infinity.

Problem 8. (10 points) *For a system $x' = y$, $y' = ax + by$ we have $x(t) = ce^{-kt}\cos(\omega t)$ for certain positive values of c, k, and ω. Which of the pictures below is most likely to represent the corresponding trajectory in the* phase plane*? Please mark the direction of increasing time on the trajectory. You must give a short explanation for your choice.*

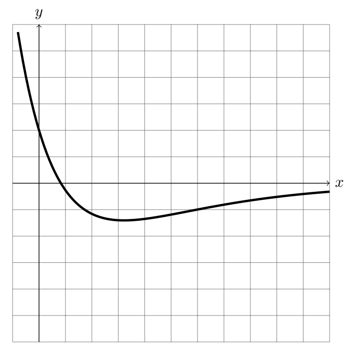
(i)



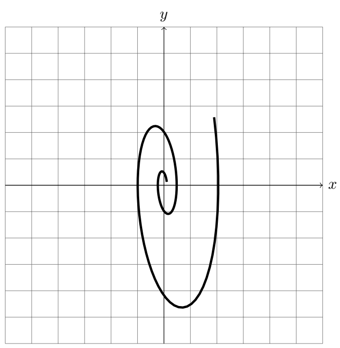
(ii)



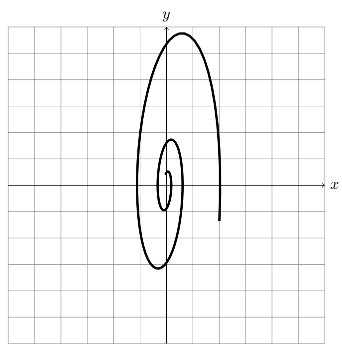
(iii)



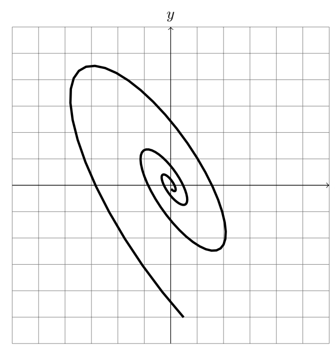
(iv)



(v)



(vi)



Solution: In the phase plane, this is a spiral sink: We see this because e^{-kt} goes to 0, so both x and y go to 0 as $t \to \infty$ and the factor of $\cos(\omega t)$ tell us the trajectory spirals.

The spiral turns clockwise. We can see this in a number of ways.

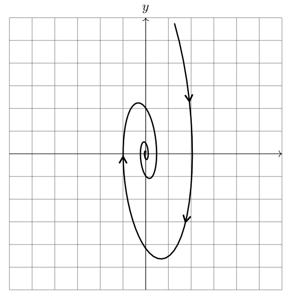
Way 1: The 1 in the upper right entry of the coefficient matrix tells us that the tangent vector at $(0, 1)$ points right, so the spiral turns clockwise.

Way 2: The trajectory moves to the right when $y = x' > 0$ and to the left when $y = x' < 0$. That is, it must be clockwise around the origin.

Way 3: The coefficient matrix is $\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$. Since the characteristic roots are $-k \pm \omega$, we must have tr$A < 0$ and $\det A > 0$, i.e., $a < 0$, $b < 0$. Since $a < 0$, the spiral turns clockwise.

Finally, on the x-axis, the tangent vector is $\mathbf{x}' = \begin{bmatrix} 0 \\ a \end{bmatrix}$, i.e., it is vertical.

Only (iv) spirals clockwise into the origin and is vertical as it crosses the x-axis. Here is the plot with arrows:



Units 2: Problems 9,10

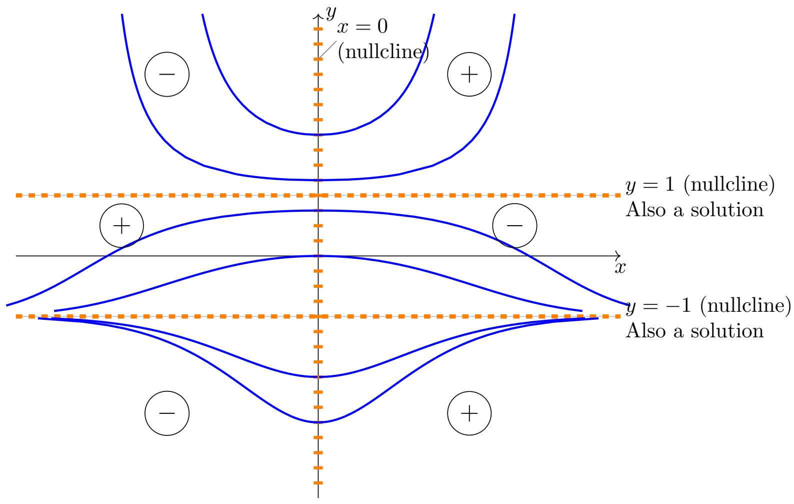
Problem 9. (20 points) *Consider the equation* $y' = xy^2 - x$

(a) [15] *(i) Draw the nullclines.* You don't need any more isoclines.

(ii) The nullclines divide the plane into regions. Mark each region where the direction field has positive slope by + with a circle around it and each region of negative slope by a − with a circle around it.

(iii) Using just what you have done in Parts (i) and (ii) sketch some representative solutions of the equation (including straight line solutions).

Nullclines: $y' = x(y^2 - 1) = 0 \quad \longrightarrow x = 0$ or $y = \pm 1$.



(b) [5] *If $y(0) = a$, then for which values of a does $y(x)$ tend to a finite value as $x \to \infty$? (Consider cases and specify the limiting value in each case.)*

Solution: We have the following table:

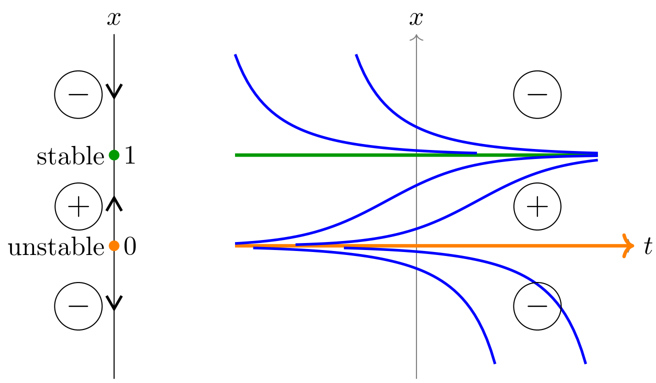
$a = 1$	$y(x) = 1$	(so $y(x) \longrightarrow 1$)
$-1 < a < 1$	$y(x) \longrightarrow -1$	
$a = -1$	$y(x) = -1$	(so $y(x) \longrightarrow -1$)
$a < -1$	$y(x) \longrightarrow -1$	
$a > 1$	$y(x) \longrightarrow \infty$	

So y tends to a finite value for $a \leq 1$. The limits for different values of a are in the table.

Problem 10. (25 points)
Let $x(t)$ represent the fraction of a population which is infected with a certain disease. This disease has a cure rate b and we model the net rate of change of $x(t)$ by $x' = -bx + x(1-x)$.

(a) [10] *Take $b = 0$, and find and classify the critical points, draw a phase line diagram and a sketch of some representative solutions.*

Solution: Critical points: $x' = x(x-1) = 0 \longrightarrow x = 0, 1$.

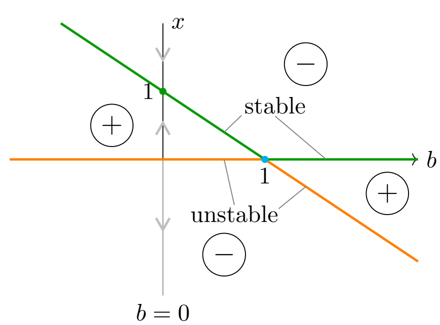


Phaseline Solution curves

(b) [10] *Now let $b > 0$ be arbitrary and draw the bifurcation diagram. Be sure to label the stable and unstable branches of the diagram.*

Solution: Critical points: $x' = -bx + x(1-x) = 0 \longrightarrow x(-b+1-x) = 0 \longrightarrow x = 0$, or $b = 1 - x$.

From Part (a) we have arrows in every region except 1 (see phase line drawn on bifurcation diagram). This gives us the sign of x' in these regions –indicated by the large pluses and minuses. To find the sign of x' in the last region, we take b large and x small and negative, say $b = 100$, $x = -1$. This gives $x' = 98$, which shows $x' > 0$ in that region. Using the pluses and minuses, we can label and color code the stable and unstable branches.



(c) [5] *Give the range of b for which the disease dies out and the range for which it becomes endemic (doesn't die out) at a positive fraction of the population. For the endemic case, give the fraction in terms of b.*

Solution: The disease dies out if $b \geq 1$ and becomes endemic if $0 \leq b < 1$.

For the endemic case, the fraction sick will be $\boxed{1-b}$ (the critical value).

Unit 3: Problems 11,12

Problem 11. (20 points)
A certain 2×2 matrix A has the following two properties:

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}3\\1\end{bmatrix} = -\begin{bmatrix}6\\2\end{bmatrix}.$$

(a) [5] *What are the eigenvectors and eigenvalues of A?*

Solution: Eigenpairs: $\lambda = 1$, $\mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\lambda = -2$, $\mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix}$.

(b) [10] *What is A (explicitly: work out the four entries).*

Solution: $S = \begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}$, $\Lambda = \begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$, $S^{-1} = \begin{bmatrix}-1/2 & 3/2\\1/2 & -1/2\end{bmatrix}$.

Diagonalization: $A = S\Lambda S^{-1} = \begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}-1/2 & 3/2\\1/2 & -1/2\end{bmatrix} = \begin{bmatrix}-7/2 & 9/2\\-3/2 & 5/2\end{bmatrix}$.

(c) [5] *Write down a diagonalization of A^5 (but you don't need to multiply out).*

Solution: $A^5 = S\Lambda^5 S^{-1} = \begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & (-2)^5\end{bmatrix}\begin{bmatrix}-1/2 & 3/2\\1/2 & -1/2\end{bmatrix}$

Problem 12. (30 points)
A is a certain 3×4 matrix but we only know its first and third columns:

$$A = \begin{bmatrix}1 & \boxed{} & 4 & \boxed{}\\2 & \boxed{} & 5 & \boxed{}\\3 & \boxed{} & 6 & \boxed{}\end{bmatrix}$$

(a) [10] *The reduced echelon form of A is $R = \begin{bmatrix}1 & 2 & 0 & 3\\0 & 0 & 1 & 4\\0 & 0 & 0 & 0\end{bmatrix}$. Fill in the missing entries in A.*

Solution: The pivot columns in R are Columns 1 and 3. We see that

$$\text{Col}_2 = 2\text{Col}_1 \quad \text{ and } \text{Col}_4 = 3\text{Col}_1 + 4\text{Col}_3.$$

We use the same relations for Columns 2 and 4 in A. So,

$$A = \begin{bmatrix} 1 & 2 & 4 & 19 \\ 2 & 4 & 5 & 26 \\ 3 & 6 & 6 & 33 \end{bmatrix}.$$

(b) [10] *Write down a basis for the null space of A.*

Solution: Working with R we have

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & -4 & 1 \end{matrix}$$

So a basis of Null(A) contains the two vectors

$$\mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v_2} = \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

(c) [10] *For what value of a does $A\mathbf{x} = \begin{bmatrix} 3 \\ 3 \\ a \end{bmatrix}$ have a solution?*

Find a solution in that case.

Solution: We can solve the system if $\begin{bmatrix} 3 \\ 3 \\ a \end{bmatrix}$ is in the column space of A. Since the free columns are redundant, this is the same as solving $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ a \end{bmatrix}$.

Row reduction gets us to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ a-3 \end{bmatrix}.$$

The bottom row represents the equation $0 = a - 3$, so we need $\boxed{a = 3.}$. Setting the free variables to 0, we get a solution $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Unit 4: Problems 13,14

Problem 13. (15 points)
Solve $2x'' + 5x' + 3x = 3\delta(t)$ with rest initial conditions.

Solution: Pre-IC: $x(0^-) = 0$, $x'(0^-) = 0$

Post-IC: $x(0^+) = 0$, $x'(0^+) = x'(0^-) + 3/2 = 3/2$

For $t < 0$: $x(t) = 0$.

For $t > 0$: $2x'' + 5x' + 3x = 0$, $x(0^+) = 0$, $x(0^+) = 3/2$

Characteristic equation: $2r^2 + 5r + 3 = 0$; roots: $\dfrac{-5 \pm \sqrt{25-24}}{4} = -\dfrac{3}{2}, -1$.

General homogeneous solution: $x(t) = c_1 e^{-3t/2} + c_2 e^{-t}$.

Use the post-IC to find c_1 and c_2.

$$x(0^+) = c_1 + c_2 = 0$$
$$x'(0^+) = -\frac{3}{2}c_1 - c_2 = \frac{3}{2}.$$

Solving, we get $c_1 = -3$, $c_2 = 3$. So,

$$x(t) = \begin{cases} 0 & \text{for } t < 0. \quad \text{(Okay if this line is missing.)} \\ -3e^{-3t/2} + 3e^{-t} & \text{for } t > 0 \end{cases}$$

Problem 14. (25 points)

Let $f(t) = \displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2(n+1)} \sin(2nt)$.

(a) [5] *What is the smallest period of the function $f(t)$?*

Solution: When $n = 1$, $\sin(2t)$ has $\boxed{\text{periond } \pi.}$

(b) [10] *Find a particular solution to the DE $x'' + x' + 4x = f(t)$ in series form.*

Solution: Solve in pieces: $x_n'' + x_n' + 4x_n = \sin(2nt)$.

The SRF gives $x_{n,p}(t) = \dfrac{\sin(2nt - \phi(n))}{|P(2in)|}$, where $\phi(n) = \text{Arg}(P(2in))$.

We have $P(2in) = 4 - 4n^2 + 2in$. So, $|P(2in)| = \sqrt{(4-4n^2)^2 + 4n^2}$ and

$\boxed{\phi(n) = \text{Arg}(P(2in)) = \tan^{-1}(2n/(4-4n^2)) \text{ in Q1 or Q2.}}$

(More precisely, for $n = 1$, we have $\phi(n) = \pi/2$ and for $n > 1$, we have $\phi(n)$ is in Q2.)

Thus, $x_{n,p}(t) = \dfrac{\sin(2nt - \phi(n))}{\sqrt{(4-4n^2)^2 + 4n^2}}$.

By superpostion $\boxed{x_p(t) = \displaystyle\sum_{n=1}^{\infty} \frac{4x_{n,p}}{n^2(n+1)} = \sum_{n=1}^{\infty} \frac{4\sin(2nt - \phi(n))}{n^2(n+1)\sqrt{(4-4n^2)^2 + 4n^2}}.}$

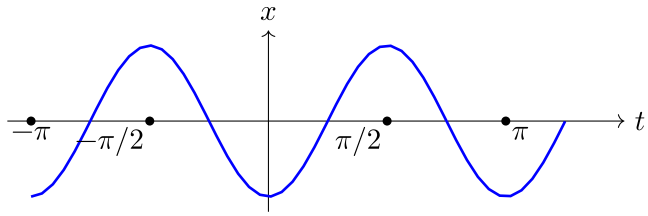
(c) [5] *Give a rough sketch of the solution $x(t)$ found in Part (b), and explain how you know that it is close to the actual graph of $x(t)$.*

Solution: When $n = 1$ the coefficient of $\sin(2t - \phi(1))$ is 1.

When $n = 2$ the coefficient of $\sin(4t - \phi(2))$ is $\dfrac{4}{12\sqrt{12^2 + 16}} \approx \dfrac{1}{36}$.

When $n = 3$ the coefficient of $\sin(6t - \phi(3))$ is $\dfrac{4}{36\sqrt{32^2 + 36}} \approx \dfrac{1}{9 \cdot 32}$.

Since the coefficients after $n = 1$ are small and decay fast, the graph looks a lot like the first term in the series. We have $\phi(1) = \text{Arg}(P(2i)) = \text{Arg}(2i) = \pi/2$. So the first term is $\sin(2t - \pi/2)$ (which equals $-\cos(2t)$).



(d) [5] *Give the Fourier series for $f'(t)$*

Solution: Differentiate term-by-term $\boxed{f'(t) = \sum_{n=1}^{\infty} \frac{8}{n(n+1)} \cos(2nt).}$

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