Name ____________________

ES.1803 Quiz 7, Spring 2024

No books, notes or calculators.

There is a formula sheet on the last pages of the quiz.

Please work neatly. Most of your work should be on the blank paper we have provided. Do not squeeze it into the margins of the printed exam.

*****Please put your name on every page you turn in.*****

1. You must do Problems 1-4 which cover Unit 1.

2. You must do Problems 5-8 which cover Unit 5 (the last unit in the class).

3. You need to pick the problems from one other unit to do. If you try problems from more than one unit, you must indicate clearly which unit you want us to grade.

Unit 1: 1. ______ 2. ______ 3. ______ 4. ______ Total: ______

Unit 5: 5. ______ 6. ______ 7. ______ 8. ______ Total: ______

Unit 2: 9. ______ 10. ______ Total: ______

Unit 3: 11. ______ 12. ______ Total: ______

Unit 4: 13. ______ 14. ______ Total: ______

Unit 1: Problems 1-4. Required

Problem 1. (15 points)
(a) [10] Find the general solution to $x'' - 100x = 5$.

(b) [5] A certain constant coefficient operator $P(D)$ is such that all solutions to $P(D)x = 0$ are linear combinations of functions of the form e^{-t} times sinusoids of angular frequencies 1 and 2.

What is the characteristic polynomial $P(r)$?

For Part (b), you should assume that its leading coefficient is 1*. You may leave your answer as a product, but you must eliminate any complex numbers that appear.*

Problem 2. (40 points)
(a) [10] Find the sinusoidal solution of $\dfrac{d^3x}{dt^3} + 2\dfrac{dx}{dt} + x = 5\cos(3t)$

Express your answer in the form $A\cos(\omega t - \phi)$.

(b) [10] Find a real-valued solution to $x'' + x = \cos(t)$.

(c) [10] Find a real-valued solution to $x'' + 2x' + 5x = e^t \cos(2t)$.

(d) [10] All we know about a certain linear time invariant operator $P(D)$ is that

$$P(D)\sin(3t) = \cos(3t).$$

Find a solution to $P(D)x = 4\cos(3t - \frac{\pi}{5})$.

Problem 3. (10 points) In this problem m and k are positive constants.

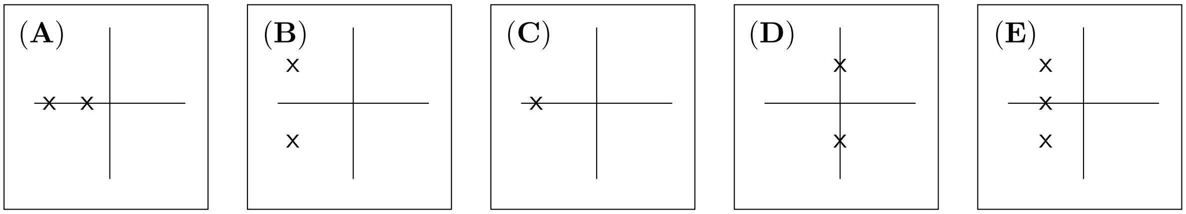
(a) [5] What type of physical system is modeled by the DE $x' + k\,x = 0$?

(We're looking for a short answer to this.)

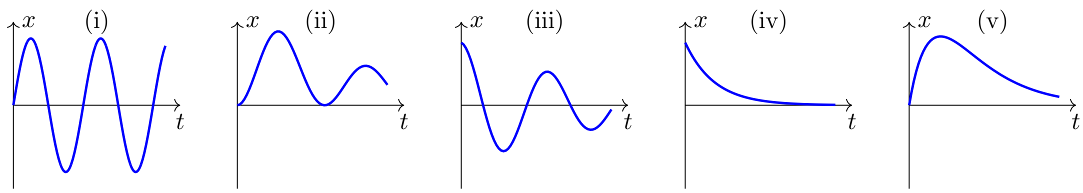
(b) [5] What type of physical system is modeled by the DE $m\ddot{x} + k\,x = 0$?

(We're looking for a short answer to this.)

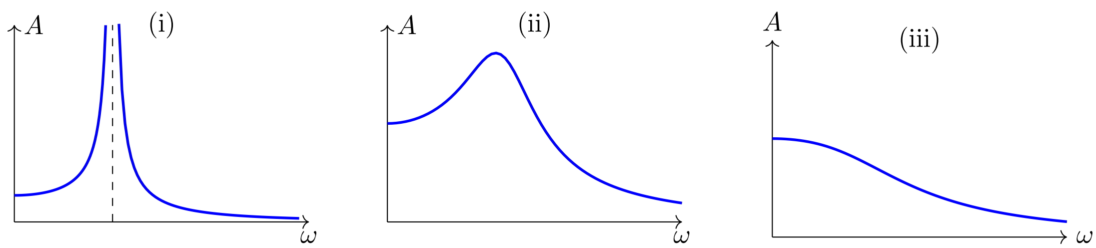
Problem 4. (16 points)
Each of the following plots are in the complex plane and give the pole diagram for a linear time invariant system of the form $P(D)x = f$



(a) [10] Below are graphs of solutions to $P(D)x = 0$ for each of the five systems. Using the labels A-E, label each graph with the letter of the corresponding pole diagram.



(b) [6] Below are graphs of the amplitude responses of the systems A, B and D to inputs of the form $f(t) = \cos(\omega t)$. Using the labels match the amplitude response to the correct pole diagram.



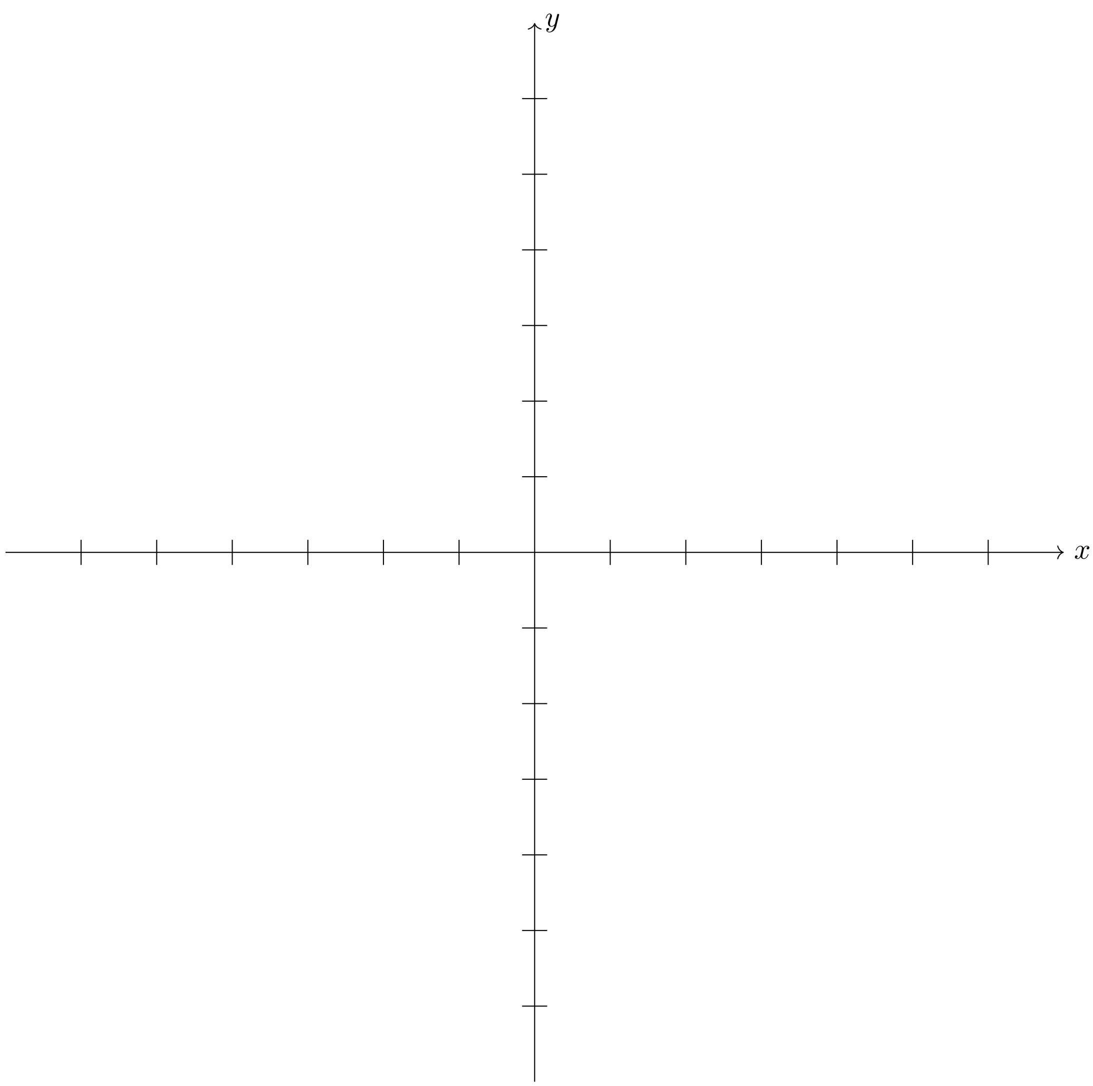
Unit 5: Problems 5-8. Required

Problem 5. (25 points)
The DE system $x' = xy - 18, \quad y' = x - 2y$ has critical points (-6,-3) and (6,3).

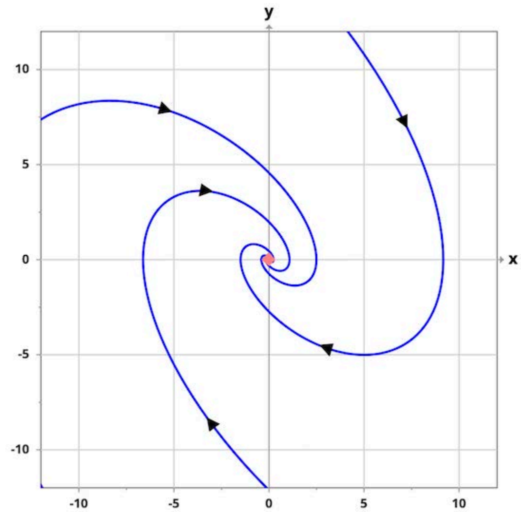
(a) [10] Compute the linearized system at each of the critical points, solve for the eigenvalues and give the type of linearized critical point. Solve for the eigenvectors only if they will be needed in order to get a good sketch of the trajectories in Part (c).

(b) [5] Will the behavior of the trajectories of the non-linear system near the critical points be essentially the same as that of the linearized system in each case? What property of the linearized system at the critical point allows you to be able to tell in each case?

(c) [10] Using the information about the linearized system found in Parts (a) and (b), sketch in (on the axes below) some trajectories in the neighborhood of each critical point. Then use this to create a conjectural picture of the trajectories of the non-linear system.



Problem 6. (15 points)
The phase plane portrait shown represents some trajectories for a DE system $\mathbf{x}' = A\mathbf{x}$. Here $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$, a and b are constants, and $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

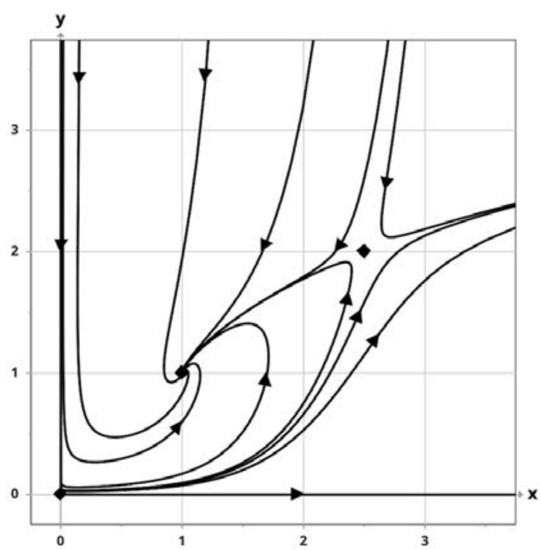


(a) [10] Using the given phase plane picture, sketch the graph of $x(t)$ corresponding to the solution to the system with IC $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(That is, sketch x vs. t, where $x(t)$ is the first component of a solution $\mathbf{x}(t)$).

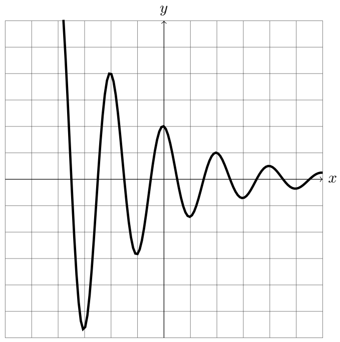
(b) [5] Give a formula expressing y in terms of x.

Problem 7. (10 points)
The phase portrait shown is for a non-linear system relating the populations of two interacting species. Describe the possible long-term behavior of the two populations.

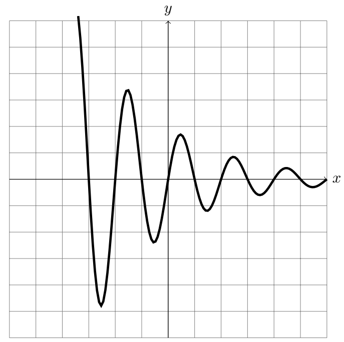


Problem 8. (10 points) For a system $x' = y$, $y' = ax + by$ we have $x(t) = ce^{-kt}\cos(\omega t)$ for certain positive values of c, k, and ω. Which of the pictures below is most likely to represent the corresponding trajectory in the *phase plane*? Please mark the direction of increasing time on the trajectory. You must give a short explanation for your choice.

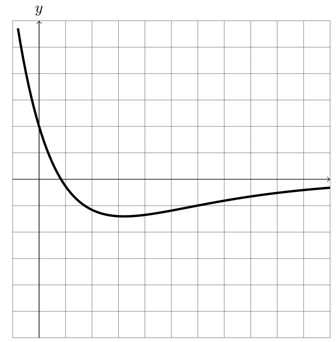
(i)



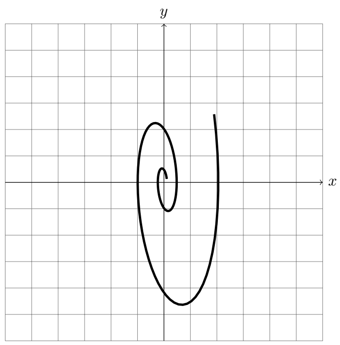
(ii)



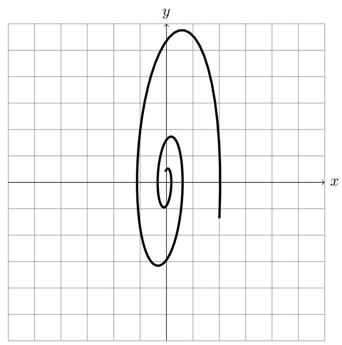
(iii)



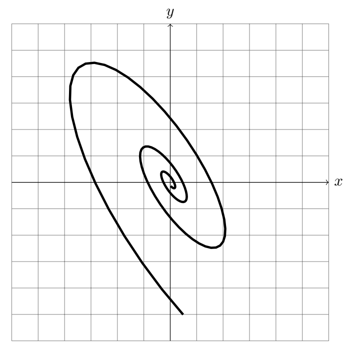
(iv)



(v)



(vi)



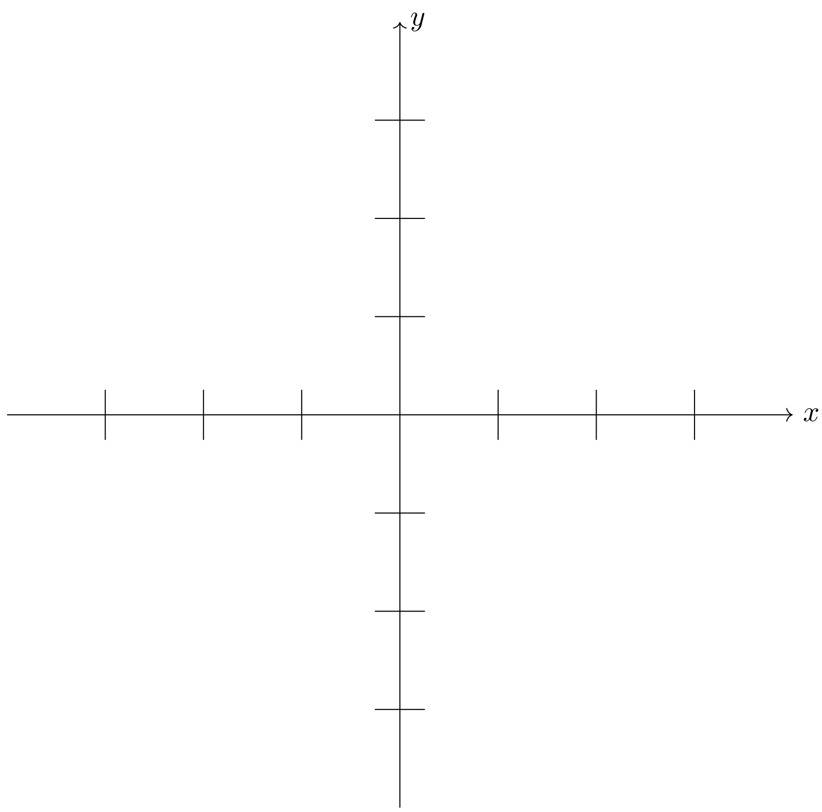
Units 2: Problems 9,10

Problem 9. (20 points) Consider the equation $y' = xy^2 - x$

(a) [15] (i) Draw the nullclines. *You don't need any more isoclines.*

(ii) The nullclines divide the plane into regions. Mark each region where the direction field has positive slope by + with a circle around it and each region of negative slope by a − with a circle around it.

(iii) Using just what you have done in Parts (i) and (ii) sketch some representative solutions of the equation (including straight line solutions).



(b) [5] If $y(0) = a$, then for which values of a does $y(x)$ tend to a finite value as $x \to \infty$? (Consider cases and specify the limiting value in each case.)

Problem 10. (25 points)
Let $x(t)$ represent the fraction of a population which is infected with a certain disease. This disease has a cure rate b and we model the net rate of change of $x(t)$ by $x' = -bx + x(1-x)$.

(a) [10] Take $b = 0$, and find and classify the critical points, draw a phase line diagram and a sketch of some representative solutions.

(b) [10] Now let $b > 0$ be arbitrary and draw the bifurcation diagram. Be sure to label the stable and unstable branches of the diagram.

(c) [5] Give the range of b for which the disease dies out and the range for which it becomes endemic (doesn't die out) at a positive fraction of the population. For the endemic case, give the fraction in terms of b.

Unit 3: Problems 11,12

Problem 11. (20 points)
A certain 2×2 matrix A has the following two properties:

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}3\\1\end{bmatrix} = -\begin{bmatrix}6\\2\end{bmatrix}.$$

(a) [5] What are the eigenvectors and eigenvalues of A?

(b) [10] What is A (explicitly: work out the four entries).

(c) [5] Write down a diagonalization of A^5 (but you don't need to multiply out).

Problem 12. (30 points)
A is a certain 3×4 matrix but we only know its first and third columns:

$$A = \begin{bmatrix} 1 & \boxed{} & 4 & \boxed{} \\ 2 & \boxed{} & 5 & \boxed{} \\ 3 & \boxed{} & 6 & \boxed{} \end{bmatrix}$$

(a) [10] The reduced echelon form of A is $R = \begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$. Fill in the missing entries in A.

(b) [10] Write down a basis for the null space of A.

(c) [10] For what value of a does $A\mathbf{x} = \begin{bmatrix}3\\3\\a\end{bmatrix}$ have a solution?

Find a solution in that case.

Unit 4: Problems 13,14

Problem 13. (15 points)
Solve $2x'' + 5x' + 3x = 3\delta(t)$ with rest initial conditions.

Problem 14. (25 points)
Let $f(t) = \displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2(n+1)} \sin(2nt)$.

(a) [5] What is the smallest period of the function $f(t)$?

(b) [10] Find a particular solution to the DE $x'' + x' + 4x = f(t)$ in series form.

(c) [5] Give a rough sketch of the solution $x(t)$ found in Part (b), and explain how you know that it is close to the actual graph of $x(t)$.

(d) [5] Give the Fourier series for $f'(t)$

Formula Sheet for Final Quiz

Variation of parameters formula

$$x(t) = x_h(t) \int \frac{q(t)}{x_h(t)}\, dt + C x_h(t).$$

Fourier Series for $f(t)$, periodic with period $2L$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t)\, \cos\left(\frac{\pi}{L} nt\right)\, dt, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t)\, \sin\left(\frac{\pi}{L} nt\right)\, dt$$

Period 2π square wave:

$$\mathrm{sq}(t) = \begin{cases} -1 \text{ on } -\pi < t < 0 \\ 1 \text{ on } 0 < t < \pi \end{cases} \qquad = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$$

Period 2π triangle wave:

$$\mathrm{tri}(t) = |t| \text{ on } -\pi < t < \pi \ = \ \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}$$

Exponential and Sinusoidal Response Formulas

$P(D)y = e^{at}$ has particular solution:

$$y_p(t) = \begin{cases} \dfrac{e^{at}}{P(a)} & \text{if } P(a) \neq 0 \\[2ex] \dfrac{te^{at}}{P'(a)} & \text{if } P(a) = 0 \text{ and } P'(a) \neq 0 \end{cases}$$

$P(D)y = \cos(\omega t)$ has particular solution

$$y_p(t) = \begin{cases} \dfrac{\cos(\omega t - \phi)}{|P(i\omega)|} & \text{where } \phi = \mathrm{Arg}(P(i\omega)) \quad \text{if } |P(i\omega)| \neq 0 \\[2ex] \dfrac{t\cos(\omega t - \phi)}{|P'(i\omega)|} & \text{where } \phi = \mathrm{Arg}(P'(i\omega)) \quad \text{if } P(i\omega) = 0 \text{ and } P'(i\omega) \neq 0 \end{cases}$$

(Integral table on next page)

Integrals

For n a positive integrer

$$\begin{aligned}
\int_0^\pi t\sin(nt)\,dt &= \frac{\pi(-1)^{n+1}}{n} \text{ if } n \neq 0. \\
\int_0^\pi t\cos(nt)\,dt &= \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases} \\
\int_0^\pi t^2\sin(nt)\,dt &= \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases} \\
\int_0^\pi t^2\cos(nt)\,dt &= \frac{2\pi(-1)^n}{n^2} \text{ if } n \neq 0.
\end{aligned}$$

For arbitrary ω

$$\begin{aligned}
\int t\sin(\omega t)\,dt &= \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}. \\
\int t\cos(\omega t)\,dt &= \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}. \\
\int t^2\sin(\omega t)\,dt &= \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}. \\
\int t^2\cos(\omega t)\,dt &= \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.
\end{aligned}$$