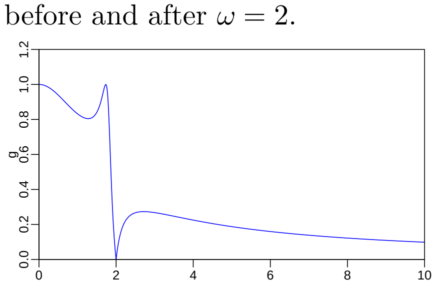
Differential Equations Review Sheet 3, Spring 2024

I. Amplitude, Phase Lag, Resonance

1. Consider the system
$$my'' + by' + ky = kF_0\cos(\omega t),$$
where **we have declared** $F_0\cos(\omega t)$ to be the input.
   - The characteristic polynomial is $P(r) = mr^2 + br + k$.
   - The input (angular) frequency is ω.
   - The periodic solution (response) is $y_p(t) = g(\omega)F_0\cos(\omega t - \phi(\omega))$.
   - The natural frequency of the system is $\omega_0 = \sqrt{k/m}$. This is the frequency of oscillation of the undamped unforced spring: $mx'' + kx = 0$.
   - $A = g(\omega)F_0$ is called the amplitude, where $g(\omega) = k/|P(i\omega)|$.
     The function $g(\omega)$ is called the gain or amplitude response of the system. It depends on ω (and m, b and k).
   - $\phi(\omega)$ also depends on ω. The function $\phi(\omega)$ is called the phase lag or the phase response of the system.
   - Practical resonance occurs if $g(\omega)$ has a maximum value at ω_r (for $\omega_r > 0$).
     If there is no such maximum, then the system does not have practical resonance.
   - Pure resonance occurs when the denominator in $g(\omega)$ is zero. We then say ω is a pure resonant frequency and the gain is infinite. For our second order system, this only happens when $b = 0$. In this case, the natural frequency ω_0 is a pure resonant frequency.
   - In the second order case with $b = 0$, the response when $w = \omega_0$ is $y_p(t) = \dfrac{t\sin\omega_0 t}{2m\omega_0}$.
     This is not a sinusoid, rather it is a 'growing' oscillation.
2. Remember the gain depends on what we consider the input. For example, consider the DE: $my'' + by' + ky = bF_0\cos(\omega t)'$, but still consider $F_0\cos(\omega t)$ to be the input. Then the gain is $g(\omega) = \dfrac{b\omega}{|P(i\omega)|}$. There are many variations on this.
3. The same language applies to higher order systems. For example, consider the system
$$x''' + x'' + 3x' + 4x = y'' + 4y,$$
where we consider y to be the input. If we let $y = B\cos(wt)$, then we find the gain is
$$g(w) = \frac{|4-\omega^2|}{|P(iw)|} = \frac{|4-\omega^2|}{\sqrt{(4-\omega^2)^2 + (3\omega-\omega^3)^2}}.$$
Here's a plot of the gain. There are no pure resonant frequencies. We see practical resonant frequencies just before and after $\omega = 2$.



End of review sheet 3