Differential Equations Review Sheet 4, Spring 2024

I. Basic Linear algebra

1. Linearity
   - For matrices and vectors: $A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1 A\mathbf{v_1} + c_2 A\mathbf{v_2}$.
   - For DEs: $P(D)(c_1 x_1 + c_2 x_2) = c_1 P(D)x_1 + c_2 P(D)x_2$.
   - The general solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.
2. Vector spaces
   - Closed under addition and scalar multiplication.
     - Examples are $\mathbf{R}^2$, $\mathbf{R}^n$, spaces of functions, the set of solutions to homegeneous linear equations.
   - Subspace: any subset that is also a vector space.
   - Basis of a vector space: a set of independent vectors that span the space.
   - Dimension of a vector space: the size of any basis.
3. Vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$
   - span of $\mathbf{v_1}, \ldots, \mathbf{v_n}$ = set of all linear combinations.
   - Independence: vectors are independent if there are no linear relations between the vectors, i.e., no nontrivial linear combination is zero.
   - Testing for independence
     - For two vectors: check that they aren't multiples of each other
     - For n vectors: put them in the columns of a matrix and check that the matrix has rank n.
4. Matrix multiplication
   - Linear combination of columns

     Example: $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.
   - Rows times columns (what you learned in 18.02)
   - Block matrices: As long as the blocks are compatibly sized

     $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} AE + BF \\ CE + DF \end{bmatrix}$$
5. Row reduction
   - Why do we use it?

     It helps us solve systems, find null spaces, find inverses, test if vectors are independent.
   - Echelon form
     - It has pivots. There are 0's below the pivots.
   - Reduced row echelon form (RREF)
     - The pivots are all 1.
     - It has 0's above and below the pivots.
     - The all 0 rows are at the bottom.
   - Pivot variables and free variables
     - Pivot variables go with pivot columns.
     - Free variables go with the other columns.
   - Rank: the number of pivots
   - Null Space: all the solutions to $A\mathbf{x} = \mathbf{0}$.
     - A and its RREF have the *same* null space

  - Dimension of Null(A) = number of free variables of A
  - Finding a basis
    For each free variable: set it to 1 and the other free variables to 0.
    Then solve for the pivot variables.
    This is easy using the RREF.
- Column space: the span of the columns
  - All $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ can be solved
  - Basis: the pivot columns of A (you use the RREF to know which columns are pivot columns)
  - Dimension = rank of the matrix = number of pivot variables
- Solving $A\mathbf{x} = \mathbf{b}$: Augment A with $\mathbf{b}$ and use row reduction to simplify the system of equations.

6. Inverse,transpose, determinant
   - Multiplication
     - $(AB)^{-1} = B^{-1}A^{-1}$
     - $(AB)^T = B^T A^T$
     - $\det(AB) = \det(A)\det(B)$
   - Use of A^{-1} in solving $A\mathbf{x} = \mathbf{b}$.

I. Eigenvalues and eigenvectors

1. Eigenvalues and eigenvectors
   - Definition: $A\mathbf{v} = \lambda\mathbf{v}$ (for some nonzero vector $\mathbf{v}$).
   - Computing:
     First find λ using the characteristic equation: $|A - \lambda I| = 0$.
     Then find the corresponding $\mathbf{v}$ by finding a basis of Null($A - \lambda I$).
     For 2×2 matrices you can find $\mathbf{v}$ by inspection. For bigger matrices you need to use row reduction to find the null space of $A - \lambda I$.
   - Eigenspaces are null spaces:
     - The eigenspace for eigenvalue λ is Null($A - \lambda I$).
     - If Null(A) is nontrivial, then 0 is an eigenvalue and Null(A) is the corresponding eigenspace.
   - Diagonalization: $A = S\Lambda S^{-1}$
   - Powers
     - $A^n = S\Lambda^n S^{-1}$
     - $e^{At} = Se^{\Lambda t}S^{-1}$ (Not done this year.)

II. Systems of DEs

1. What are they?
   Systems with more than one dependent variable.
2. How do I solve a system of DEs?
   - Eigenvalue method
     - When do I use it?
       For constant coefficient systems of the form $\mathbf{x}' = A\mathbf{x}$.
     - General solution: For the system $\mathbf{x}' = A\mathbf{x}$, assume λ_1, λ_2, ...are the eigenvalues of A with corresponding basic eigenvector $\mathbf{v_1}$, $\mathbf{v_2}$, ... Then, the solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2} + \ldots$$

- Modal solutions (normal modes): $\mathbf{x_1}(t) = e^{\lambda_1 t}\mathbf{v_1}$, $\mathbf{x_2}(t) = e^{\lambda_2 t}\mathbf{v_2}$ etc.
- What if I get complex eigenvalues?
  Just like when you got complex roots of your characteristic polynomial, pick one root $\lambda_1 = a + bi$, find its eigenvector and let $\mathbf{z}(t) = e^{\lambda_1 t}\mathbf{v_1}$. Take the real and imaginary parts of $\mathbf{z}(t)$ to get a pair of independent solutions.
- General solution using the matrix exponential: $\mathbf{x} = e^{At}\mathbf{x}(\mathbf{0})$ (Not done this year.)

• Decoupling: This is a change of variables using $A = S\Lambda S^{-1}$
  - Change variables: $\mathbf{x} = S\mathbf{u}$
  - New system: $\mathbf{u}' = \Lambda\mathbf{u}$.

3. The companion system (anti-elimination)
   • What is it?
     - The companion system to a higher order DE is an equivalent system of first-order equations.
   • How do I do find the companion system?
     - Start with a higher order DE and introduce new variables to produce a system.
   • Example: $x'' - 8x' + 7x = 0$. Let $x_1 = x$ and $x_2 = x'$ $\Rightarrow$ $x_2' - 8x_2 + 7x_1 = 0$ $\Rightarrow$ system is $x_1' = x_2$, and $x_2' = -7x_1 + 8x_2$.

End of review sheet 4