18.03: Substitution methods

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1 Introduction

These notes are left over from a time when substitution methods was on the syllabus. They give a brief explanation and some examples of several methods for solving first-order DEs.

2 Various methods

2.1 Straight substitution

To solve a DE of the form $\dfrac{dy}{dx} = F(ax + by + c)$, make the following substitution.

$$v = ax + by + c \qquad \text{(Remember } y \text{ is a function of } x.\text{)}$$

This implies: $\dfrac{dv}{dx} = a + b\dfrac{dy}{dx} = a + bF(v)$. This is a separable DE.

Example 1. Solve $\dfrac{dy}{dx} = (x + 9y + 7)^2$.

Solution: Let $v = x + 9y + 7 \quad \Rightarrow \dfrac{dv}{dx} = 1 + 9\dfrac{dy}{dx} = 1 + 9v^2$.

Separating variables: $\displaystyle\int \frac{dv}{1 + 9v^2} = \int dx$.

Integrating: $\frac{1}{3}\tan^{-1} 3v = x + C \quad \Rightarrow v = \frac{1}{3}\tan(3x + C)$.

Back substitute: $x + 9y + 7 = \dfrac{1}{3}\tan(3x + C) \quad \Rightarrow y = \dfrac{1}{9}\left(\dfrac{1}{3}\tan(3x + C) - x - 7\right)$

Note: y is only defined on an interval, e.g., if $C = 0$ then it's only defined for $-\pi/2 < 3x < \pi/2$.

Example 2. Solve $\dfrac{dy}{dx} = (x + y + 1)^5$.

Solution: Let $v = x + y + 1 \quad \Rightarrow \dfrac{dv}{dx} = 1 + v^5$.

Separate variables: $\displaystyle\int \frac{dv}{1 + v^5} = \int dx$. This is painful, but doable as an integral.

2.2 Homogeneous equations

Homogeneous equations have the form $y' = f(y/x)$.

Note: This is a different use of the word homogeneous than in our constant coefficient, homogeneous, linear differential equations.

Substitution: $w = y/x$.

This implies: $y = xw$, so $y' = w + xw' = f(w) \quad \Rightarrow w' = f(w)/x - w/x$.

This is a separable equation!

Example 3. Show $y' = \dfrac{x^3 + 3xy^2 + y^3}{x^2y + xy^2}$ is homogeneous.

Solution: Note, both numerator and denominator are made up of terms of degree 3.

Divide top and bottom by x^3: $y' = \dfrac{1 + 3(y/x)^2 + (y/x)^3}{y/x + (y/x)^2}$, which has the form, $y' = f(y/x)$.

Example 4. Solve $y' = \dfrac{xy}{x^2 - y^2}$ (prime means derivative with respect to x).

Solution: Divide top and bottom by x^2: $y' = \dfrac{y/x}{1 - (y/x)^2}$.

Let $w = y/x$. So, $wx = y \quad \Rightarrow w'x + w = y'$. Thus,

$$w'x + w = \frac{w}{1 - w^2} \qquad \Rightarrow x\frac{dw}{dx} = \frac{w}{1 - w^2} - w = \frac{w^3}{1 - w^2}.$$

This is separable.

Separate variables: $\displaystyle\int \frac{1 - w^2}{w^3}\, dw = \int \frac{1}{x}\, dx$.

Do the integration: $-\dfrac{1}{2w^2} - \ln w = \ln x + C$.

We have an implicit solution. We won't try to solve for w as an explicit function of x.

Example 5. Solve $y' = \dfrac{x^{3/2}y + y^{1/2}x^2}{x^{5/2}}$.

Solution: We rewrite the right-hand side to get: $y' = y/x + (y/x)^{1/2}$.

The substition $w = y/x$ leads to the DE: $\dfrac{dw}{dx} = \dfrac{w^{1/2}}{x}$.

Then, separation of variables gives

$$\int \frac{dw}{w^{1/2}} = \int \frac{dx}{x} \qquad \Rightarrow 2w^{1/2} = \ln x + C \qquad \Rightarrow w = \left(\frac{1}{2}\ln x + C\right)^2.$$

(Note: in the last equation, we changed the meaning of C.)

Back substition: $y = x\left(\dfrac{1}{2}\ln x + C\right)^2$.

2.3 Bernoulli Equations

Bernoulli equations have the form: $\dfrac{dy}{dx} + p(x)y = q(x)y^n$.

This almost looks like a linear DE. We can convert it to a linear first-order DE as follows.

Algebra: $y^{-n}y' + py^{1-n} = q$.

Substitute: $w = y^{1-n} \quad \Rightarrow w' = (1-n)y^{-n}y'$. So,

$$\frac{1}{1-n}w' + pw = q \qquad \text{Linear DE.}$$

Note: This works as long as $n \neq 1$. But, in that case the DE is already linear.

Example 6. Solve $x^2y' + xy + y^2 = 0$.

Solution: Put in standard form: $y' + \frac{1}{x}y = -\frac{1}{x^2}y^2$.

This implies: $y^{-2}y' + \frac{1}{x}y^{-1} = -\frac{1}{x^2}$.

Substitution: Let $w = y^{-1} \quad \Rightarrow w' = -y^{-2}y' \quad \Rightarrow -w' + \frac{1}{x}w = -\frac{1}{x^2} \quad \Rightarrow w' - \frac{1}{x}w = \frac{1}{x^2}$.

This is a linear, first-order DE, which we can solve using the variation of parameters formula.

Homogeneous solution: $w_h(x) = e^{-\int -\frac{1}{x}\,dx} = e^{\ln x} = x$.

Variation of parameters: $w = x\left(\int \frac{1}{x^3} + C_1\right) = x\left(-\frac{1}{2x^2} + C_1\right) = \frac{-1 + Cx^2}{2x}$.

Back substitute: $y = \frac{1}{w} = \frac{2x}{Cx^2 - 1}$

Note: This equation also happens to be homogeneous.

2.4 Double Substitution

Example 7. Solve $\frac{dy}{dx} = \frac{x+y}{3+x+2y}$.

Solution: Double substitution: Let $u = x + y, \quad v = 3 + x + 2y$

Consider u to bethe independent variable, i.e., $v = v(u)$

A little algebra shows

$$x = 2u - v + 3,\ y = -u + v - 3 \quad \Rightarrow \frac{dx}{du} = 2 - \frac{dv}{du}, \quad \frac{dy}{du} = -1 + \frac{dv}{du}.$$

Now,

$$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{2 - dv/du}{-1 + dv/du} = \frac{u}{v}.$$

This leads to the homogeneous DE: $\frac{dv}{du} = \frac{2u+v}{u+v}$.

2.5 'Flipping'

Sometime just turning the equation over makes things easier.

Example 8. Solve $\dfrac{dx}{dt} = \dfrac{1}{x^3 - tx^2}$.

Solution: Take the reciprocal of both sides:

$$\frac{dt}{dx} = x^3 - x^2 t.$$

If we consder t to be the dependent and x the independent variable, we have a first-order linear equation. We won't go through the integrals to solve this.

End of substitution methods notes