ES.1803 Topic 1 Notes

Jeremy Orloff

1 Introduction to differential equations

1.1 Goals

1. Know the definition of a differential equation.
2. Know our first and second most important equations and their solutions.
3. Be able to derive the differential equation modeling a physical or geometric situation.
4. Be able to solve a separable differential equation, including finding lost solutions.
5. Be able to solve an initial value problem (IVP) by solving the differential equation and using the initial condition to find the constant of integration.

1.2 Differential equations and solutions

A differential equation (DE) is an equation with derivatives!

Example 1.1. (DEs modeling physical processes, i.e., rate equations)

1. Newton's law of cooling: $\frac{dT}{dt} = -k(T - A)$, where T is the temperature of a body in an environment with ambient temperature A.

2. Gravity near the earth's surface: $m\frac{d^2x}{dt^2} = -mg$, where x is the height of a mass m above the surface of the earth.

3. Hooke's law: $m\frac{d^2x}{dt^2} = -kx$, where x is the displacement from equilibrium of a spring with spring constant k.

Other examples: Below we will give some examples of differential equations modeling some geometric situations.

A solution to a differential equation is any function that satisfies the DE. Let's focus on what this means by contrasting it with solving an algebraic equation.

The unknown in an algebraic equation, such as

$$y^2 + 2y + 1 = 0$$

is the number y. The equation is solved by finding a numerical value for y that satisfies the equation. You can check by substitution that $y = -1$ is a solution to the equation shown.

The unknown in the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

is the **function** $y(x)$.The equation is solved by finding a function $y(x)$ that satisfies the equation One solution to the equation shown is $y(x) = e^{-x}$. You can check this by substituting $y(x) = e^{-x}$ into the equation. Again, note that the **solution is a function**.

More often we will say that *the* solution is a family of functions, e.g., $y = Ce^{-t}$. The parameter C is like the constant of integration in 18.01. Every value of C gives a different function which solves the DE.

1.3 The most important differential equation in 18.03

Here, in the very first class, we state and give solutions to our most important differential equations. In this case we will check the solutions by substitution. As we proceed in the course we will learn methods that help us discover solutions to equations.

The most important DE we will study is

$$\frac{dy}{dt} = ay, \tag{1}$$

where a is a constant (in units of 1/time). In words the equation says that

the rate of change of y is proportional to y.

Because of its importance we will write down some other ways you might see it:

$$y' = ay; \qquad \frac{dy}{dt} = ay(t); \qquad y' - ay = 0; \qquad \dot{y} - ay = 0.$$

In the last equation, we used the physicist 'dot' notation to indicate the derivative is with respect to time. You should recognize that all of these are the same equation.

The solution to this equation is

$$y(t) = Ce^{at},$$

where C is any constant.

1.3.1 Checking the solution by substitution

The above solution is easily checked by substitution. Because this equation is so important we show the details. Substituting $y(t) = Ce^{at}$ into Equation 1 we have:

$$\begin{aligned} \text{Left side of 1:} \quad & y' = aCe^{at} \\ \text{Right side of 1:} \quad & ay = aCe^{at} \end{aligned}$$

Since after substitution the left side equals the right, we have shown that $y(t) = Ce^{at}$ is indeed a solution of Equation 1.

1.3.2 The physical model of the most important DE

As a physical model this equation says that the quantity y changes at a rate proportional to y.

Because of the form the solution takes we say that Equation 1 models exponential growth or decay.'

In this course we will learn many techniques for solving differential equations. We will test almost all of them on Equation 1. After learning these techniques, you should, of course, understand how to use them to solve 1. However: whenever you see this equation you should remind yourself that it models exponential growth or decay and you should know the solution without computation.

1.4 The second most important differential equation

Our second most important DE is

$$my'' + ky = 0, \tag{2}$$

where m and k are constants. You can easily check that, with $\omega = \sqrt{k/m}$, the function

$$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

is a solution. Equation 2 models a simple harmonic oscillator. More prosaically, it models a mass m oscillating at the end of a spring with spring constant k.

1.5 Solving differential equations by the method of optimism

In our first and second most important equations above we simply told you the solution. Once you have a possible solution it is easy to check it by substitution into the differential equation. We will call this method, where you guess a solution and check it by plugging your guess into the equation, the method of optimism. In all seriousness, this will be an important method for us. Of course, its utility depends on learning how to make good guesses!

1.6 General form of a differential equation

We can always rearrange a differential equation so that the right hand side is 0. For example, $y' = ay$ can be written as $y' - ay = 0$. With this in mind the most general form for a differential equation is

$$F(t, y, y', \ldots, y^{(n)}) = 0,$$

where F is a function. For example,

$$(y')^2 + e^{y'' \sin(t)} - y^{(4)} = 0.$$

The order of a differential equation is the order of the highest derivative that occurs. So the example just above shows a DE of order 4.

1.7 Constructing a differential equation to model a physical situation

We use rate equations, i.e., differential equations, to model systems that undergo change. The following argument using Δt should be somewhat familiar from calculus.

Example 1.2. Suppose a population $P(t)$ has constant birth and death rates:

$$\beta = 2\%/\text{year}, \;\; \delta = 1\%/\text{year}$$

Build a differential equation that models this situation.

Solution: In the interval $[t, t + \Delta t]$, the change in P is given by

$$\Delta P = \text{ number of births - number of deaths.}$$

Over a small time interval Δt the population is roughly constant so:

$$\text{Births in the time interval} \approx P(t) \cdot \beta \cdot \Delta t$$
$$\text{Deaths in the time interval} \approx P(t) \cdot \delta \cdot \Delta t$$

Combining these we have: $\Delta P \approx P(t)\,\beta\,\Delta t - P(t)\,\delta\,\Delta t$. So,

$$\frac{\Delta P}{\Delta t} \approx (\beta - \delta)P(t).$$

Finally, letting Δt go to 0 we have derived the differential equation

$$\frac{dP}{dt} = (\beta - \delta)P.$$

Notice that if $\beta > \delta$ then the population is increasing.

Of course, this DE is our most important DE 1: the equation of exponential growth or decay. We know the solution is $P = P_0 e^{(\beta-\delta)t}$.

Note: Suppose β and δ are more complicated and depend on t, say $\beta = P + 2t$ and $\delta = P/t$. The derivation of the DE is the same, i.e.

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P = (P + 2t - P/t)P.$$

Because β and δ are no longer constants, this is not a situation of exponential growth and the solution will be more complicated (and probably harder to find).

Example 1.3. Bacteria growth. Suppose a population of bacteria is modeled by the exponential growth equation $P' = kP$. Suppose that the population doubles every 3 hours. Find the growth constant k.

Solution: The equation $P' = kP$ has solution $P(t) = Ce^{kt}$. From the initial condition we have that $P(0) = C$. Since the population doubles every 3 hours we have $P(3) = Ce^{3k} = 2C$.

Solving for k we get $\boxed{k = \frac{1}{3}\ln 2 \quad \text{(in units of 1/hours.)}}$

1.8 Initial value problems

An initial value problem (IVP) is just a differential equation where one value of the solution function is specified. We illustrate with some simple examples.

Example 1.4. Initial value problem. Solve the IVP $\dot{y} = 3y$, $y(0) = 7$.

Solution: We recognize this as an exponential growth equation, so $y(t) = Ce^{3t}$. Using the initial condition we have $y(0) = 7 = C$. Therefore, $\boxed{y(t) = 7e^{3t}.}$

Example 1.5. Initial value problem. Solve the IVP $y' = x^2$, $y(2) = 7$.

Solution: Note, the use of x indicates that the independent variable in this problem is x. This is really an 18.01 problem: integrating we get $y = x^3/3 + C$. Using the initial condition we find $C = 7 - 8/3$.

1.9 Separable Equations

Now it's time to learn our first technique for solving differential equations. A first-order DE is called separable if the variables can be separated from each other. We illustrate with a series of examples.

Example 1.6. Exponential growth. Use separation of variables to solve the exponential growth equation $y' = 4y$.

Solution: We rewrite the equation as $\frac{dy}{dt} = 4y$. Next we separate the variables by getting all the y's on one side and the t's on the other.

$$\frac{dy}{y} = 4\,dt.$$

Now we integrate both sides:

$$\int \frac{dy}{y} = \int 4\,dt \quad \Leftrightarrow \quad \ln|y| = 4t + C.$$

Now we solve for y by exponentiating both sides:

$$|y| = e^C e^{4t} \quad \textit{or} \quad y = \pm e^C e^{4t}.$$

Since $\pm e^C$ is just a constant we rename it simply K. We now have the solution we knew we'd get:

$$y = Ke^{4t}.$$

Example 1.7. Here is a standard example where the solution goes to infinity in a finite time (i.e., the solutions 'blow up'). One of the fun features of differential equations is how very simple equations can have very surprising behavior.

Solve the initial value problem

$$\frac{dy}{dt} = y^2; \qquad y(0) = 1.$$

Solution: We can separate the variables by moving all the y's to one side and the t's to the other

$$\frac{dy}{y^2} = dt$$

Integrating both sides we get: $-\frac{1}{y} = t + C$

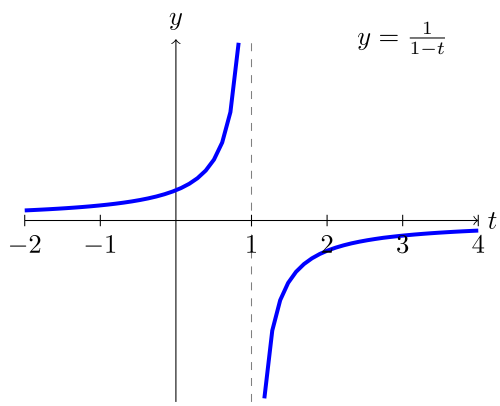
Think: The constant of integration is important, but we only need it on one side.

Solving for y we get the solution:

$$y = -\frac{1}{t+C}.$$

Finally, we use initial condition $y(0) = 1$ to find that $C = -1$. So the solution is: $\boxed{y(t) = \frac{1}{1-t}}$.

We graph this function below. Note that the graph has a vertical asymptote at $t = 1$.



Graph of the function $1/(1-t)$

1.9.1 Technical definition of a solution

Looking at the previous example we see the domain of y consists of two intervals: $(-\infty, 1)$ and $(1, \infty)$. For technical reasons we will require that the domain of a solution consists of exactly one interval. So the above graph really shows **two solutions**:

Solution 1: $y(t) = 1/(1-t)$, where y is in the interval $(-\infty, 1)$
Solution 2: $y(t) = 1/(1-t)$, where y is in the interval $(1, \infty)$

In the example problem, since our IVP had $y(0) = 1$ the solution must have $t = 0$ in its domain. Therefore, solution 1 is the solution to the example's IVP.

1.9.2 Lost solutions

We have to cover one more detail of separable equations. Sometimes solutions get *lost* and have to be recovered. This is a small detail, but you want to pay attention since it's worth 1 easy point on exams and psets.

Example 1.8. In the example $y' = y^2$, we found the solution $y = -\frac{1}{t+C}$. But it is easy to check by substitution that $y(t) = 0$ is also a solution. Since this solution can not be written as $y = -1/(t + C)$ we call it a lost solution.

The simple explanation is that it got lost when we divided by y^2. After all if $y = 0$ it was not legitimate to divide by y^2.

General idea of lost solutions for separable DEs

Suppose we have the differential equation

$$y' = f(x)g(y)$$

If $g(y_0) = 0$ then you can check by substitution that $y(x) = y_0$ is a solution to the DE. It may get lost in when we separate variables because dividing by by $g(y)$ would then mean dividing by 0.

Example 1.9. Find all the (possible) lost solutions of $y' = x(y-2)(y-3)$.

Solution: In this case $g(y) = (y-2)(y-3)$. The lost solutions are found by finding all the roots of $g(y)$. That is, the lost solutions are $y(x) = 2$ and $y(x) = 3$.

1.9.3 Implicit solutions

Sometimes solving for y as a function of x is too hard, so we don't!

Example 1.10. Implicit solutions. Solve $y' = \frac{x^3+3x+1}{y^6+y+1}$.

Solution: This is separable and after separating variables and integrating we have

$$\frac{y^7}{7} + \frac{y^2}{2} + y = \frac{x^4}{4} + \frac{3x^2}{2} + x + C.$$

This is too hard to solve for y as a function of x so we leave our answer in this implicit form.

1.9.4 More examples

Example 1.11. Solve $\dfrac{dy}{dx} = xy$.

Solution: Separating variables: $\dfrac{dy}{y} = x\,dx$. Therefore, $\displaystyle\int \frac{dy}{y} = \int x\,dx$, which implies $\ln y = \dfrac{x^2}{2} + C$. Finally after exponentiation and replacing e^C by K we have $\boxed{y = Ke^{x^2/2}.}$

Think: There is a lost solution that was found by some sloppy algebra. Can you spot the solution and the sloppy algebra?

Example 1.12. Solve $\dfrac{dy}{dx} = x^3y^2$.

Solution: Separating variables and integrating gives: $-\frac{1}{y} = \frac{x^4}{4} + C$. Solving for y we have

$$y = -\frac{4}{x^4 + 4C}.$$

There is also a lost solution: $y(x) = 0$.

Example 1.13. Solve $y' + p(x)y = 0$.

Solution: We first rewrite this so that it's clearly separable: $\frac{dy}{y} = -p(x)\,dx$. After the usual separation and integration we have

$$\log(|y|) = -\int p(x)\,dx + C$$

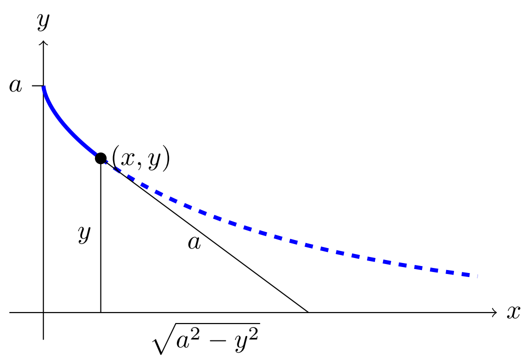
Therefore, $|y(x)| = e^C e^{-\int p(x)\,dx}$ and $y(x) = 0$ is a lost solution.

1.10 Geometric Applications of DEs

Since the slope of a curve is given by its derivatives, we can often use differential equations to describe curves.

Example 1.14. An heavy object is dragged through the sand by rope. Suppose the object starts at $(0, a)$ with the puller at the origin, so the rope has length a. The puller moves along the x-axis so that the rope is always taut and tangent to the curve followed by the object. This curve is called a tractrix. Find an equation for it.

Solution: Since the rope is tangent to the curve, its slope is $\frac{dy}{dx}$. Also, computing the slope geometrically as rise/run, the diagram below shows that $\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}$.



The tractrix

Thus, $-\frac{\sqrt{a^2 - y^2}}{y}\,dy = dx$. Integrating (details below) we get

$$a \ln\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right) - \sqrt{a^2 - y^2} = x + C.$$

The initial position $(x, y) = (0, a)$ implies $C = 0$. Therefore, $\boxed{x = a \ln\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right) - \sqrt{a^2 - y^2}.}$

To finish the problem, we show that the integral is what we claimed it was:

Let $I = -\int \frac{\sqrt{a^2 - y^2}}{y}\,dy$.

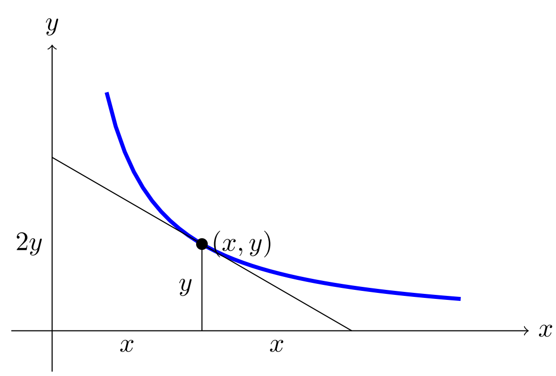
Now use the trig. substitution: $y = a\sin u$:

$$\begin{aligned}\Rightarrow I &= -\int \frac{a\cos u}{a\sin u} a\cos u\,du = -a\int \frac{\cos^2 u}{\sin u}\,du \\ &= -a\int \frac{1-\sin^2 u}{\sin u}\,du = -a\int \csc u - \sin u\,du \\ &= a\ln(\csc u + \cot u) - a\cos u\end{aligned}$$

Back substituting we get $I = -\sqrt{a^2-y^2} + a\ln\left(\frac{a+\sqrt{a^2-y^2}}{y}\right)$, which is what we claimed above.

Example 1.15. Suppose $y = y(x)$ is a curve in the first quadrant and that the part of the curve's tangent line that lies in the first quadrant is bisected by the point of tangency. Find and solve the DE for this curve.

Solution: The figure shows the piece of the tangent bisected by the point (x, y) on the curve. Thus the slope of the tangent $= \frac{dy}{dx} = \frac{-y}{x}$. This differential equation is separable and is easily solved: $\boxed{y = C/x.}$

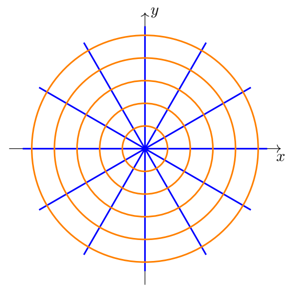


1.11 Orthogonal trajectories

This is mostly taken from the 18.03 Supplementary Notes by Arthur Mattuck.

Given a one-parameter family of plane curves, its orthogonal trajectories are another one-parameter family of curves, each one of which is perpendicular to all the curves in the original family.

Example 1.16. Take the family consisting of all circles having center at the origin, i.e., the one-parameter family of curves $x^2 + y^2 = c^2$. We know that all the rays from the origin are orthogonal to all the circles. That is the orthogonal trajectories to the circles are all the rays (half-lines) starting at the origin.



Blue rays are orthogonal to orange circles wherever they meet.

The examples below will show how to find orthogonal trajectories using differential equations.

Orthogonal trajectories arise in different contexts in applications. For example, if the original family represents the lines of force in a gravitational or electrostatic field, its orthogonal trajectories represent the equipotentials, the curves along which the gravitational or electrostatic potential is constant.

To find the orthogonal trajectories for a one-parameter family:

1. Find the ODE $y' = f(x, y)$ satisfied by the family.

2. The orthogonal family has DE $y' = -\dfrac{1}{f((x, y)}$. That is, the solutions of this DE are the orthogonal trajectories to the original family.

This works because at any point (x, y), the original curve has slope $f(x, y)$, so the orthogonal curve must have slope $-1/f(x, y)$ (negative reciprocal).

Example 1.17. Find the orthogonal trajectories to the family of curves $y = c\,x^n$, where n is a fixed positive integer and c an arbitrary constant.

Solution: First note: If $n = 1$, the curves are lines through the origin, so the orthogonal trajectories should be the circles centered at the origin – this will help check our work.

Step 1 is to find the first-order DE of the family of curves. The parameter c cannot be in this DE – it is the parameter in the solutions.

One common trick is to isolate the c and then differentiate with respect to x. Remember when differentiating that y is a function of x.

$$y = c\,x^n \quad \xrightarrow{\text{isolate } c} \quad yx^{-n} = c \quad \xrightarrow{\text{derivative}} \quad y'x^{-n} - nyx^{-n-1} = 0.$$

Now, solving for y' gives $y' = \frac{ny}{x}$. This is the DE for our family of curves.

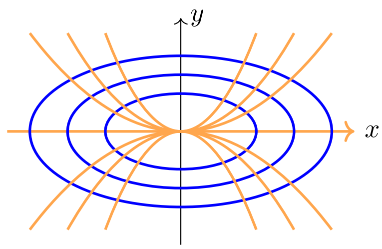
The DE for the orthogonal trajectories is then

$$y' = -\frac{x}{ny}.$$

This is separable. After separating the variables and integrating, we have

$$x^2 + ny^2 = d.$$

We use d as the constant of integration because c was already used. This solution represents a family of ellipses, i.e., for each d we have the equation of an ellipse.



$n = 2$: Orthogonal families $y = cx^2$, $x^2 + 2y^2 = d$.

Note: When $n = 1$, the ellipses are circles centered at the origin, as predicted.

1.12 Definite integral solutions to IVPs

Often we can write the solution to an initial value problem using definite integrals. While *this will not play a major role in 18.03*, it can be quite useful when the integrals are hard to compute or need to be computed numerically. We illustrate with an example.

Example 1.18. Solve $y' = \sin(x^2)\cos(y^2)$, $\quad y(0) = 2$. Give the solution implicitly using definite integrals.

Solution: Seperating variables we have $\dfrac{dy}{\cos(y^2)} = \sin(x^2)\,dx$. We can write the solution as

$$\int_2^y \frac{1}{\cos(u^2)}\,du = \int_0^x \sin(v^2)\,dv.$$

Notes.

1. We used dummy variables in the integrals because x, y are in the limits.

2. The y integral starts at $y = 2$, i.e., the initial y value and the x integral starts at $x = 0$, i.e., at the initial x value.

3. Differentiating both integrals with respect to x, using the fundamental theorem of calculus and the chain rule, we get

$$\frac{1}{\cos(y^2)}\,\frac{dy}{dx} = \sin(x^2).$$

This is equivalent to the original differential equation.

4. The solution is given implicitly, i.e., a function of y = a function of x.

5. Setting $x = 0$ and $y = 2$, the integrals on both sides are 0. That is, the implicit solution satisfies the initial condition.

6. These integrals cannot be computed in terms of our usual elementary functions, but they are easily computed numerically.