

ES.1803 Topic 10 Notes

Jeremy Orloff

10 Direction fields, integral curves, existence of solutions

10.1 Goals

All of our goals refer to the first-order differential equation $y' = f(x, y)$.

1. Know the general form $y' = f(x, y)$ for a first-order DE.
2. Be able to use the method of isoclines to sketch the direction field of the DE and to sketch some integral (solution) curves.
3. Know the definition of a nullcline and be able to use nullclines to get a qualitative understanding of the solutions to a given DE.
4. Know the statement of the existence and uniqueness theorem for first-order DEs.
5. Be able to use, isoclines and known integral curves to form fences and funnels for the integral curves of a given DE.

10.2 Introduction

This unit is about first-order –not necessarily linear– differential equations. If x is the independent variable and $y(x)$ is a function of x then the [general first-order DE](#) is

$$y'(x) = f(x, y),$$

where $f(x, y)$ is some function.

Examples: $y' = x - y + 1$, $y' = x^2 + y^2$, ...

In general, it is not possible to solve first-order equations exactly. Nonetheless without solving we can find approximate numerical solutions, use visual techniques to understand the systems and determine their long-term behavior.

In this topic we will explore visualization using direction fields. We will also state a general existence and uniqueness theorem that will give us confidence that our approximate techniques are approximating something that really exists.

10.2.1 Integral curves

Here is as good a place as any to introduce the term [integral curve](#). An integral curve for a differential equation is the graph of a solution, i.e., a solution curve.

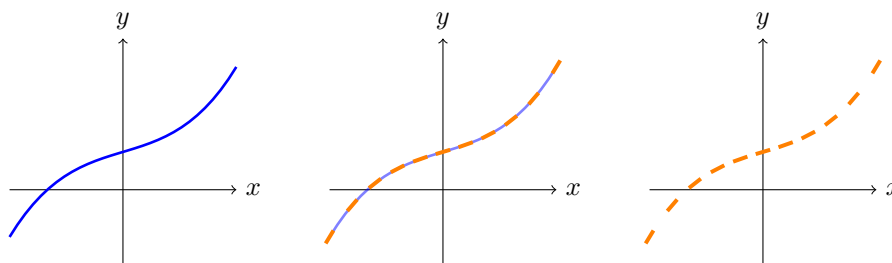
10.3 Direction or slope fields

We will motivate our use of direction fields with a simple example.

Example 10.1. Suppose you had the first-order differential equation

$$y' = f(x, y) \quad (1)$$

If you knew a solution you could simply graph it. Then at some points on the graph you could add a **direction field element**, i.e., a little tangent segment, along the graph. The first figure below shows just the curve. The second shows the curve with direction field elements added. The third figure shows just the direction field elements. Notice how well they represent the curve!



The slope elements show the shape of the curve.

We will also use the term **slope element** for direction field element.

Important point. The important point is that while we might not know the solution to Equation (1) at any point (x, y) we know the slope of the solution that goes through (x, y) , i.e., slope = $f(x, y)$. This means we can always draw the direction element at (x, y) . As we saw, these elements allow us to visualize the curves quite nicely.

10.4 Drawing direction fields using isoclines

The basic algorithm for drawing the direction field for Equation (1) is to choose a lot of points (x, y) and draw a slope element at each one. (As defined above, a slope element is a “little segment” of slope $f(x, y)$.) The key idea is that the (unknown) solution curve through (x, y) must have the same slope as the slope element.

Computer: With a computer drawing the slope field is easy, you just have the computer draw elements at an evenly spaced set grid of points. One tool we will use for this is the Isoclines mathlet: <https://mathlets.org/mathlets/isoclines/>.

By hand: People are not as patient as computers, so by hand we will use the method of isoclines. This limits the amount of computation needed and gives us some information which is not as readily accessible in the computer method.

Definition. The **isocline** of slope m for $y' = f(x, y)$ is the set of points (x, y) where $f(x, y) = m$, i.e., a set of points where all the slope elements have the same slope. (You can parse the word isocline as ‘iso = same’ and ‘cline = slope’.)

Example 10.2. (Drawing a direction field using isoclines.) Consider the initial value problem (IVP)

$$y' = \sqrt{x^2 + y^2}; \quad y(0) = 0.5.$$

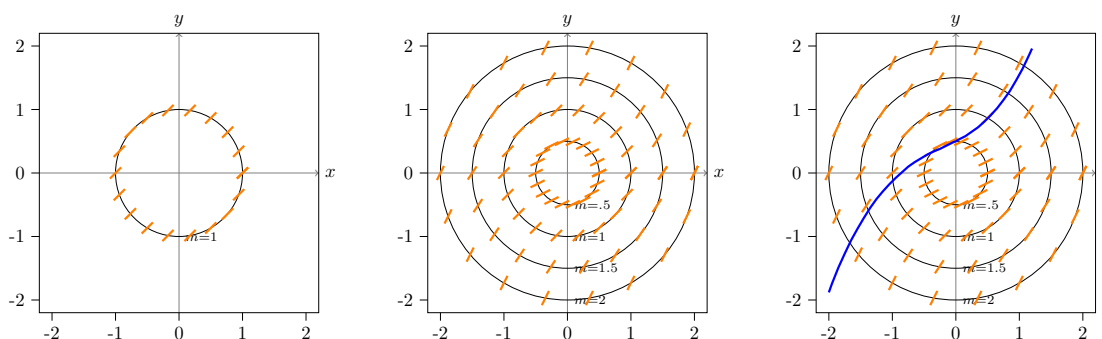
Draw a few **isoclines** ($y' = \text{constant}$) for the DE and sketch the solution curve to the IVP.

Solution: Step 1 is to draw the isoclines. We need to find the set of points where $f(x, y) = m$ for various constants m . We'll draw isoclines for $m = 0.5, 1, 1.5, 2$.

$m = 1$: In our example, the isocline $f(x, y) = 1 = \sqrt{x^2 + y^2}$ is a circle of radius 1 in the xy -plane. We plot it by drawing the circle and then adding direction field elements of slope 1 along the circle. (See first figure below.)

Likewise for $m = 0.5$ the isocline is a circle of radius $1/2$. We draw the circle and add direction field elements of slope $1/2$ along it. We repeat this for $m = 1.5$ and $m = 2$. (See second figure below.)

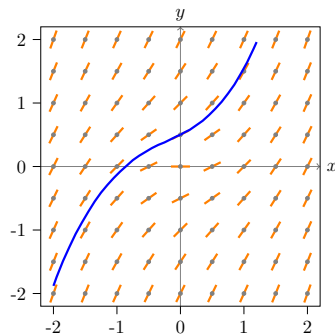
Step 2 is to sketch the solution curve $y = y(x)$ through the initial position $(0, 0.5)$. At each isocline the slope of the curve, $y'(x)$, should be the same as the slope of the direction field element on the isocline.



Isoclines and solution curve

Example 10.3. Redo the previous example using a computer to draw the slope elements at an array of points in the plane.

Solution: We instructed the computer to systematically loop through a two dimensional array of points. At each point it computes the direction element slope $f(x, y)$ and draws the element. (The integral curve was drawn using numerical methods discussed in the next topic.)



Computer generated slope field

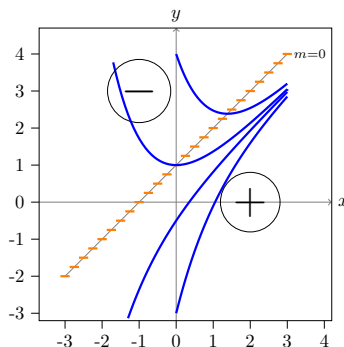
10.4.1 Nullclines

The **nullcline** for a first-order DE is the isocline corresponding to slope $m = 0$. The next example shows how just drawing the isocline can give a sense of how the solutions behave.

Example 10.4. Consider the DE $y' = f(x, y) = x - y + 1$. First draw the nullcline. Then indicate regions where the slope field has positive slope and those with negative slope. Use this information to guess at some solution curves $y = y(x)$. Describe in words how the solution curves behave.

Solution: The nullcline is where $f(x, y) = x - y + 1 = 0$, i.e., $y = x + 1$. This happens to be a line. We show it with its slope elements in the figure below. The nullcline divides the plane into two regions: above the nullcline the slope field is negative and below it, the field is positive.

With just this information, we can see that integral curves that start above the nullcline must decrease until they pass through the nullcline (with 0 slope) and then turn upwards. Those that start below the nullcline are always increasing.

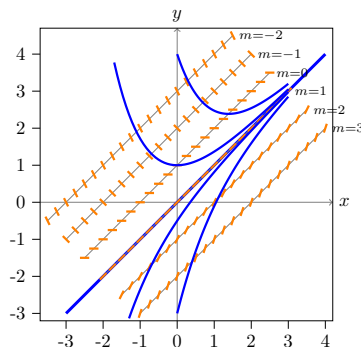


Nullcline and guessed integral curves for $y' = x - y + 1$.

Note. The existence and uniqueness theorem in the next section says that the solution curves can't cross. This means that it is a good guess –though not guaranteed– that the solution curves approach each other asymptotically as shown.

Example 10.5. Redo the previous example and include isoclines with $m = -2, -1, 0, 1, 2, 3$. Use the direction field to sketch a few solutions.

Solution: For any m the isocline $f(x, y) = m = x - y + 1$ is a line $y = x + 1 - m$. The figure shows the requested isoclines with their slope elements



It so happens (this is unusual, **don't expect it in other problems**) that the isocline for $m = 1$ is also an integral curve. All solutions go asymptotically to this curve $y = x$

Note. This example is a constant coefficient linear DE, so we could have found solutions analytically. This is certainly not the case for most first-order equations.

10.5 Existence and Uniqueness

Theorem. Existence and uniqueness for first-order differential equations.

Consider the initial value problem $y' = f(x, y)$; $y(x_0) = y_0$.

1. (Existence) If $f(x, y)$ is continuous then there is a solution.
2. (Uniqueness) If $\frac{\partial f}{\partial y}$ is also continuous then the solution is unique.

The proof of this involves more analysis than we have time for in 1803. For those who are interested, we've posted a note describing the Picard method of proof for this theorem.

Notes 1. The theorem says that if you have two different solutions $y_1(x)$ and $y_2(x)$, then for any x_0 the functions are not equal, i.e., $y_1(x_0) \neq y_2(x_0)$.

2. Graphically this means that integral curves never cross.
3. This theorem is important. It allows us to talk confidently about solutions without actually finding them.

10.5.1 Examples and counterexamples

As mathematicians it is important to remember that theorems have hypotheses and that we should check its hypotheses before using a theorem. The examples here show that the existence and uniqueness theorem can "fail" if its hypotheses are not met.

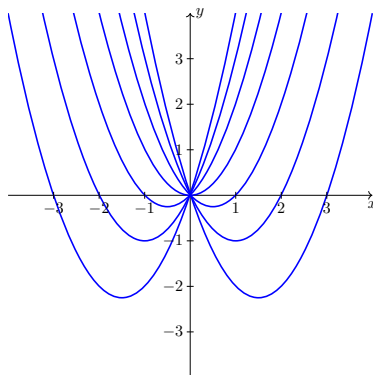
Important. Before reading these examples, remember that our main interest is in the cases where existence and uniqueness is true. Our most common application of this will be to assert that integral curves don't intersect.

Example 10.6. (Our most important DE) The IVP $y' = y$; $y(x_0) = y_0$ satisfies the hypotheses of the existence and uniqueness theorem. Therefore, it has a solution and (for different initial conditions) the integral curves don't cross.

Example 10.7. (Non-existence and non-uniqueness) (See picture.) The DE $y' = y/x + x$ doesn't satisfy the hypotheses for the existence and uniqueness theorem because $f(x, y) = y/x + x$ is not continuous at $x = 0$. In fact, uniqueness fails because all solutions satisfy the same initial condition $y(0) = 0$. This is shown in the figure below.

Proof. This is a linear equation, so, using the variation of parameters formula, we find that the general solution is $y(x) = x^2 + Cx$. All of these solutions satisfy the initial condition $y(0) = 0$.

Note, the existence part of the theorem can also fail because there are no solutions that satisfy the initial condition, e.g., $y(0) = 1$.



A case where uniqueness fails: $y' = y/x + x$

Note. Away from $x = 0$ the function $f(x, y)$ is continuous, as is $\frac{\partial f}{\partial y}$, so existence and uniqueness holds, i.e., exactly one integral curve goes through any point (x_0, y_0) as long as $x_0 \neq 0$.

Example 10.8. Here is our standard example where a solution exists and is unique, but it is only defined on an interval—not the entire number line. The IVP $y' = y^2$; $y(0) = 1$ has solution $y = \frac{1}{1-x}$.

The solution exists and is unique—and is only defined on the interval $(-\infty, 1)$.

Very briefly, here's an example where solutions always exist, but are not necessarily unique.

Example 10.9. Consider the DE $y' = 2\sqrt{|y|} = f(x, y)$

Since $f(x, y)$ continuous, the theorem says that solutions exist. For example,

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{1}{\sqrt{y}} & \text{for } y > 0 \\ -\frac{1}{\sqrt{|y|}} & \text{for } y < 0 \end{cases}$$

is *not* continuous when $y = 0$. So the existence and uniqueness theorem doesn't guarantee uniqueness. In fact, there are two solutions: $y_1(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ -x^2 & \text{for } x \leq 0 \end{cases}$, and $y_2(x) = 0$, which both have initial condition $y(0) = 0$, i.e., solutions are not unique.

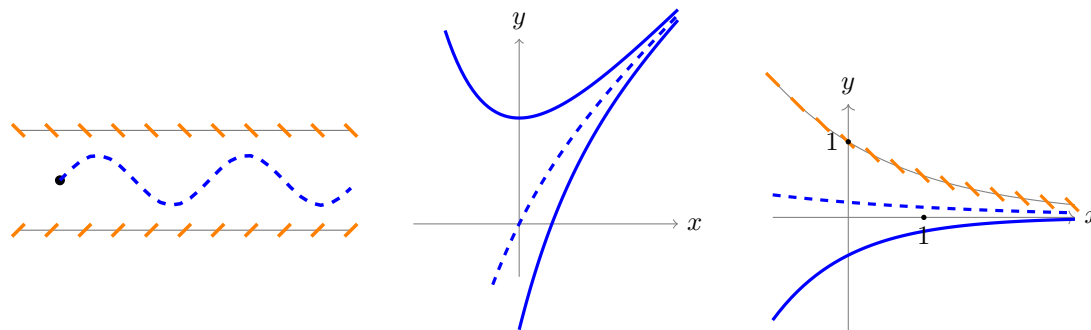
10.6 Squeezing: fences and funnels

In this section, as usual, we are looking at the first-order equation

$$y' = f(x, y).$$

To avoid problems we will assume that the existence and uniqueness theorem always holds, so that integral curves never intersect. Our goal is to see how we can use isoclines and known solutions to understand how unknown solutions will behave.

Both isoclines and integral curves can act as fences which other solution curves can't cross. Together they can form a funnel, which forces other solutions to stay between them and go asymptotically to some function. We explain this with some simple figures, which show isoclines and integral curves in several configurations.



Consider the upper isocline in the left hand figure. Since the slope field crosses from above to below this isocline, integral curves must do the same. That is, any solution that is above the isocline can cross to below, but any solution that is below the isocline must remain below it. We say, “integral curves can’t cross an isocline against the slope field”.

Since a fence is something that stops you from crossing a boundary, we call the upper isocline an **upper fence** on solutions, i.e., from below it looks like a fence to any solution. (From above, a solution does not see a fence and happily crosses it.)

Likewise, the lower isocline is a **lower fence** on solutions. That is, any solution that starts above it must stay above it.

Thus any solution, e.g., the blue dashed curve, that starts between the two fences must stay between them.

In the middle figure, all three curves are integral curves. The existence and uniqueness theorem says that integral curves can’t intersect each other. This means that integral curves act as fences (both upper and lower) for other integral curves. This is illustrated in the middle figure, where the two solid blue integral curves constrain the blue dashed integral curve to stay between them.

Notice, that in the middle figure, the two fences become asymptotically closer. This says that the blue dashed curve will be **squeezed** between the fences and become asymptotically closer to them. In this case we say that the two integral curves form a **funnel** and solutions that start between them are asymptotically the same.

In the right hand figure we have an isocline acting as an upper fence and an integral curve as a lower fence. Together they form a funnel. Just like the funnel in the middle figure any solution that starts between them is funneled between them.

Example 10.10. Look at the right hand figure. Suppose that $y(x)$ is the solution to the IVP $y' = f(x, y)$; $y(0) = 0.5$. Estimate $y(100)$.

Solution: Since the integral curve of y starts inside the funnel, it must stay there and be squeezed down to 0. Looking at the scale on the x -axis, we see that $x = 100$ is very far to the right, so $y(100) \approx 0$.

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ES.1803 Differential Equations

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