ES.1803 Topic 12 Notes

Jeremy Orloff

12 Autonomous equations and bifurcation diagrams

12.1 Goals

1. Know the standard form of an autonomous, first-order differential equation.

2. Be able to use critical points to draw the phase line for an autonomous, first-order DE.

3. Be able to draw the bifurcation diagram for an autonomous, first-order DE with a parameter.

4. Be able to interpret phase lines and bifurcation diagrams in terms of population dynamanics and sustainability.

12.2 Introduction

In this topic we look at, so-called, autonomous equations. These are a special type of nonlinear first-order equations. In general, rather than solve these equations, we will try to understand the long-term behavior of the systems they model without finding the solution.

When the system includes a parameter, we will draw bifurcation diagrams which give us a system level view of the long-term behavior of the system for all possible values of the parameter. This is analagous to our use of gain curves, which tell us, in one graph, the behavior of the system for all possible input frequencies.

The Phase Lines Mathlet `https://mathlets.org/mathlets/phase-lines/` illustrates everything we will do in this topic. We encourage you to look at it!

12.3 Autonomous differential equations

Definition. An autonomous first-order differential equation has the form

$$x'(t) = f(x).$$

(Compare this to the general first-order DE which has the form $x' = f(x, t)$.)

The word autonomous means *self-governing.* That is, x', the rate that x changes, depends only on x and not on t.

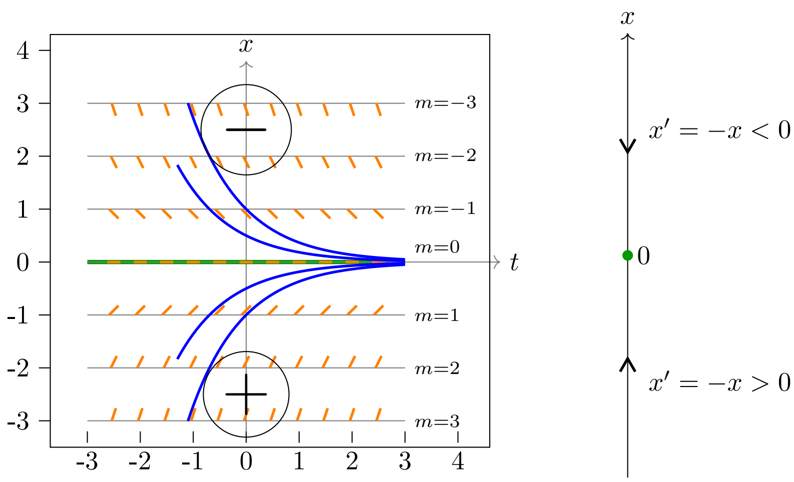
Here are some important properties of autonomous equations:
1. They are separable.
2. They can be hard to integrate.
3. We can say a lot about them without solving them. (More on this below.)
4. They are time invariant: if $x(t)$ is a solution then so is $x(t - t_0)$.

12.4 Direction fields and phase lines for autonomous equations

Our most important DE, $x' = kx$, is autonomous. We will use it to introduce phase lines for such equations. First, we look at its direction field.

Example 12.1. Use isoclines to draw the direction field for the DE $x' = -x$. Put the phase line (to be defined) next to it.

Solution: The isocline for slope m is $f(x) = -x = m$. This is a horizontal line. we draw the direction field and a few solutions using isoclines for $m = 0$, 1, 2, 3, -1, -2, -3.



Left: direction field for $\frac{dx}{dt} = -x$. Right: phase line

As always the nullcline separates the plane into regions where x' is positive and negative. These are marked with a big + and − on the direction field.

The phase line is a simplified version of the direction field. Since the direction field is independent of t, we just throw away the t-axis. The phase line is the x-axis. On it we mark the x-value of each nullcline, i.e., $x = 0$. Instead of slope field elements we put arrows indicating the direction of the slope field. These correspond to the big + and − in the direction field. In our example we have a down arrow in the region $x > 0$ and an up arrow in the region $x < 0$.

This simple example shows two important properties of autonomous equations.

1. For autonomous equations $x' = f(x)$, the isoclines are always horizontal lines. This is because the equation $f(x) = m$ is independent of t.

2. Any integral curve can be translated left or right and it is still an integral curve. That is, if $x(t)$ is a solution then so is $x(t - t_0)$. This is easy to see because the direction field is the same if you translate it right or left.

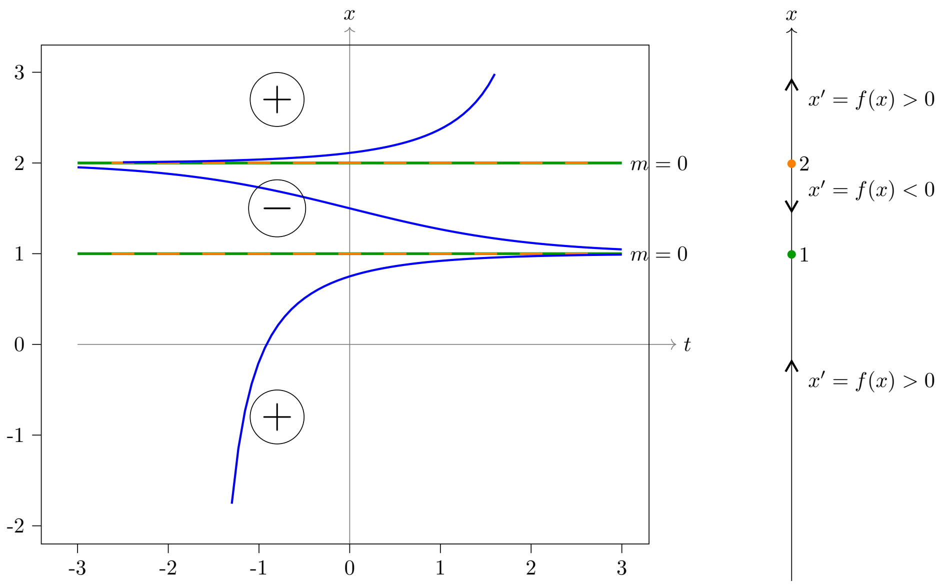
12.5 Equilibria, nullclines, constant solutions and critical points

For autonomous equations we will use a number of different words to describe nullclines. We'll introduce them through an example.

Example 12.2. Let $x' = (1 - x)(2 - x)$. Draw a direction field consisting of just the nullclines and large + or − signs indicating regions where the direction field has positive or negative slope. Using just this, sketch some solutions, including the ones along the nullclines.

Then use your direction field to draw the phase line for this system.

Solution: We have $x' = f(x) = (1-x)(2-x)$. The nullcline is where $f(x) = 0$, i.e., $x = 1$ and $x = 2$. These are horizontal lines in the tx-plane. It's easy to check that $x' > 0$ when $x > 2$ or $x < 1$ and $x' < 0$ when $1 < x < 2$. The sign of x' in different regions is marked with a + or a −.



Direction field and phase line for $x' = f(x) = (1-x)(2-x)$.

Now for the main point of this example: The nullclines $x = 1$ and $x = 2$ are clearly solutions. We use the following terms to describe them.

Because they are constants, they are called constant solutions.

Because they are unchanging, they are called equilibrium solutions.

Because $x' = 0$ along them, we call $x = 1$ and $x = 2$ critical points for the DE.

To finish the example we added solution curves. In regions where $x' > 0$ the solution curves are increasing. Because the equilibrium solutions act as fences, these solutions can't cross them. So we get the picture as shown.

The phase line is drawn next to the direction field. The arrows on the phase line show the sign of x', i.e., the direction of the slope field, for different ranges of x.

12.5.1 Lost solutions

Finally, nullclines correspond to lost solutions: The equation $\dfrac{dx}{dt} = f(x)$ is separable. When we separate variables we get $\dfrac{dx}{f(x)} = dt$. So there are lost solutions where $f(x) = 0$. These are the nullclines (or constant solutions or equilibrium solutions).

12.6 Stability of equilibria

In general, we say an equilibrium is stable if nearby solutions go asymptotically to the equilibrium value.

Example 12.3. Looking at Example 12.2, give each equilibrium and say whether it is stable or unstable.

Solution: The equilibria are the same as the constant solutions. These are $x = 1$ and $x = 2$. Looking at the phase line, we see clearly that $x = 1$ is stable and $x = 2$ is unstable. You can see the same thing in the direction field.

12.7 Analyzing an autonomous DE

We will use the following steps to analyze the autonomous DE $x' = f(x)$.

1. Find the critical points $x' = f(x) = 0$ and plot them on the phase line.

2. Determine the sign of x' for different values of x. Use these to put arrows on the phase line. This can be done algebraically or graphically.

3. Determine the stability of the equilibrium solutions.

4. If desired, sketch some solutions in the tx-plane.

We illustrate this with some examples.

Example 12.4. Let $x' = -k(x - A)$. This models Newton's law of cooling for a body of temperature x in an environment of temperature A. We assume that k and A are constants, with $k > 0$.

Plot the phase line. Be sure to indicate the stability of the equilibrium solutions. Also, give a rough sketch of solutions in the tx-plane.

Solution: First, note that this equation is simple enough that we actually know the general solution

$$x = A + ce^{-kt}.$$

You should check that our answers agree with this!

We follow the steps outlined above.

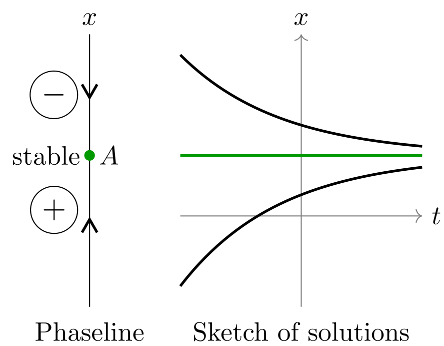
1. Find the critical points: $f(x) = -k(x - A) = 0$ implies $x = A$. This is indicated on the phase line below. Remember: For autonomous equations, critical points are the same as equilibrium solutions.

2. Determine the sign of x' for different x: This is the same algebra you used in 18.01 when graphing a function and looking for regions where it increases and decreases.

It's easy to see that when $x > A$ we have $x' = -k(x - A) < 0$. Likewise, when $x < A$ we have $x' > 0$. We use this to add arrows to the phase line. For this example, we also label regions with a $+$ or $-$.

3. The arrows on the phase line show that the equilibrium $x = A$ is stable.

4. Directly from the phase line, we can sketch some solutions. Note: these are in the tx-plane. The equilibrium solution is the horizontal line $x = A$. The other solutions are strictly qualitative: they are drawn to show that all solutions go asymptotically to the (stable) equilibrium.



Example 12.5. (Logistic equation.) Consider the autonomous system

$$x' = k(M - x)x = f(x).$$

We assume k and M are positive constants. This is called a logistic population model. For the population x it models the growth rate as $k(M - x)$. The growth rate depends on x, and decreases as x increases. (Compare this with the exponential model $x' = ax$, where the growth rate is constant.) This model captures the notion that, as the population increases, the competition for scarce resources leads to a lower growth rate. If the population gets too large the growth rate will become negative.

Plot the phase line for this system and sketch some solutions.

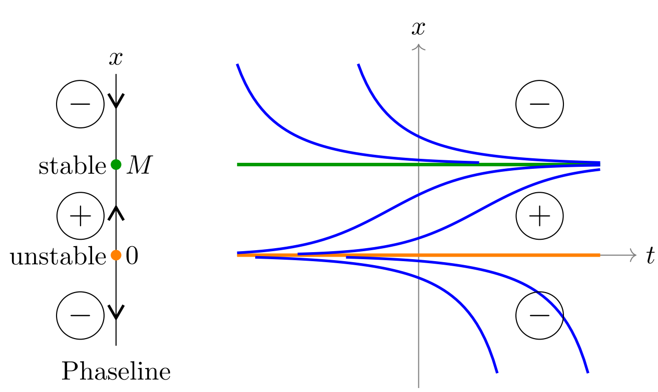
Solution: We follow the standard steps

1. Critical points: $x' = k(M - x)x = 0$ gives critical points $x = M$ or $x = 0$.

2. Looking at the x axis, it is clear we have the following signs for x':
when $x > M$, then $x' < 0$,
when $0 < x < M$, then $x' > 0$,
when $x < 0$, then $x' < 0$,

3. Using 1 and 2 we can draw the phase line. This shows that $x = M$ is a stable equilibrium and $x = 0$ is unstable.

4. Finally, it is a simple matter to sketch solution curves: they can't cross the equilibria and must go towards the stable equilibrium and away from the unstable equilibrium. As before, these are made up, but they capture the qualitative nature of the solutions.



Notes. 1. The S shaped curves between 0 and M are called logistic curves. The Wikipedia article `https://en.wikipedia.org/wiki/Logistic_function` gives a number of applications where the logistic function appears.

2. Because the population stabilizes at M and the growth rate becomes negative if $x > M$, we call M the carrying capacity of the environment.

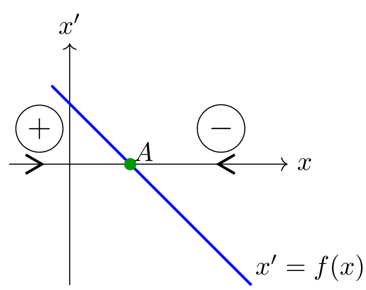
3. It is difficult to find the reason for the name logistic. The term was coined around 1844 by the French mathematician Pierre François Verhuist. (See the same Wikipedia article cited above.)

12.7.1 Graphical method for determining the sign of x'.

In the examples above we found the sign on x' by testing values in different ranges of x. Here we'll show an alternative graphical method. The trick is to graph x' vs. x. When doing this, we are viewing x' as a variable. We illustrate by redoing some of the examples.

Example 12.6. Find the phase line from Example 12.4 by graphing x' vs. x and putting the phase line on the x-axis.

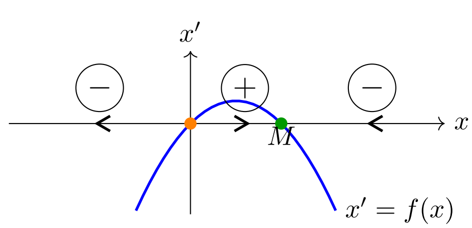
Solution: In the example we have $x' = -k(x - A)$. The graph of this is the negatively sloped line shown below. It is now easy to see the sign of x' as a function of x. When $x > A$, the graph is below the x axis, so x' is negative. Likewise, when $x < A$, the graph is above the x-axis, so x' is positive. We mark these regions with $-$ and $+$. The arrows on the x-axis correspond to these signs. Magically, the x-axis now shows the phase line for the system.



x' vs. x. The x-axis shows the phase line.

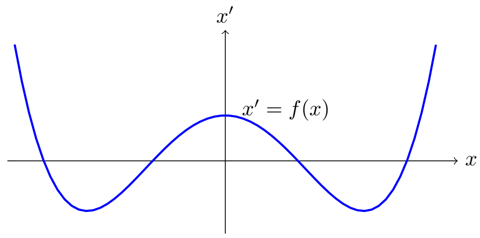
Example 12.7. Find the phase line from Example 12.5 by graphing x' vs. x and putting the phase line on the x-axis. (The DE is $x' = k(M - x)x$.)

Solution: As in the previous example we plot x' vs. x. Then we use the sign of x' to add arrows to the x-axis. The plot is a downward pointing parabola. As before, the x-axis shows the phase line.

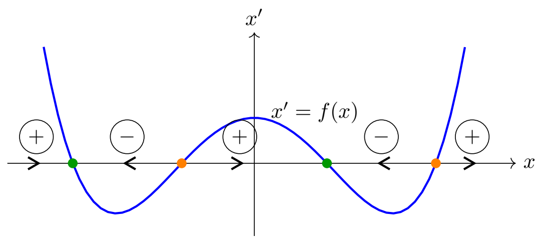


x' vs. x. The x-axis shows the phase line.

Example 12.8. The following shows the graph of $x' = f(x)$. Use the graph, to draw the phase line for this system. Indicate the critical points and their stability.



Solution: We add arrows to the graph. The critical points are marked green for stable and orange for unstable.



12.8 Parameters and bifurcation diagrams

Bifurcation diagrams help us visualize how the system behaves at different settings of a given control parameter. This is similar to what we did when we graphed gain vs. input frequency. The input frequency is a parameter and the gain curve lets us see in one figure how the system responds to any frequency.

We'll get at this idea using examples.

12.8.1 Logistic with harvesting population model

Example 12.9. This example will not show a bifurcation diagram. Instead, we will try to show how we might be led to inventing bifurcation diagrams.

Suppose you are growing irises. Left alone in your garden, the population of irises follows a logistic population model

$$x' = (3 - x)x,$$

where x is in units of 1000 irises and time is in units of months.

Your plan is to harvest and sell the flowers at a constant rate of a units/month. With this level of harvesting, the population model becomes

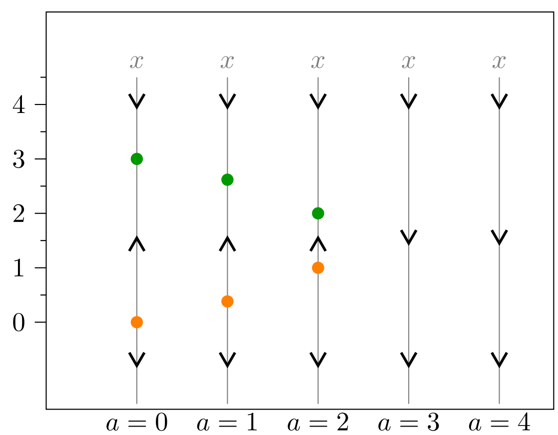
$$x' = f(x) = x(3 - x) - a. \tag{1}$$

You know that if a is too large then the iris population will crash and you'll go out of business. So your first goal to understand what happens to the population for different values of a.

For any value of a, we can draw the phase line and determine how the population will respond at that value. So your assignment for the population model in Equation 1 is to draw phase lines for every value of a!

Okay, that is probably too hard, let's just do it for each of the values $a = 0, 1, 2, 3, 4$.

Solution: It's not hard to compute critical points for each of these a. We don't show the calculation. Here are the phase lines



The phase lines for $a = 0, 1, 2$ each have two critical points. The upper one is stable and the lower one is unstable. For $a = 3, 4$, there are no critical points.

Clearly, it's a bad idea to harvest at the rates $a = 3$ or $a = 4$. In these cases the population will decrease to 0. So these rates are not sustainable.

The rates $a = 0, 1, 2$ each have a positive stable critical point. In all three cases, if we wait to start harvesting until the population is about 1.5, then the population will go to the stable critical value. This is sustainable.

Our conclusion is that it is possible to harvest at the rate $a = 2$ without ruining our business.

12.8.2 Sustainability

Definition. If the population model has a positive stable critical point we say the population is sustainable.

Note. Sustainability doesn't mean you can't mess it up. For instance, in Example 12.9, if $a = 2$ and we start harvesting when $x = 0.5$, then the population will crash to 0. This would be a bad idea, but we still say that $a = 2$ is a sustainable harvesting rate. That is, as long as you do it right and start harvesting when the population is large enough, then the population will stabilize at the stable critical point.

12.8.3 Bifurcation diagrams

In the previous example we were unable to draw phase lines for every value of a, so we drew a small number of them to help choose a harvesting rate. We saw that we could sustainably harvest when $a = 2$, but not at $a = 3$. What about other values of a? This is the motivation behind bifurcation diagrams, they'll show us how the system behaves for all values of a in one simple graph.

Definition. Suppose we have a population $x(t)$ with a model which depends on a parameter a. The bifurcation diagram for this model is the plot of all the points (a, x) in the ax-plane where the model has a critical point. We always indicate on the diagram whether the critical points represent stable or unstable equilibria.

We illustrate bifurcation diagrams by redoing Example 12.9.

Example 12.10. Draw the bifurcation diagram for the logistic with harvesting model

$$x' = x(3 - x) - a$$

For which values of a is the population sustainable?

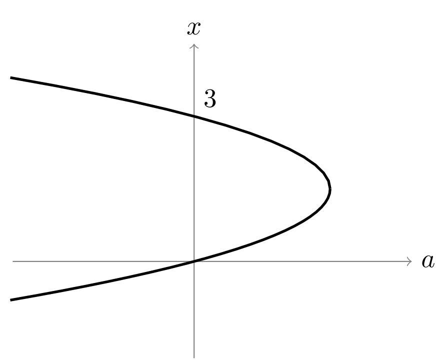
Solution: We use the following steps.

Step 1. Draw the ax-axes. Be sure to label them!

Step 2. Compute and plot all the critical points. In this case we have

$$x(3 - x) - a = 0 \quad \Rightarrow \quad a = x(3 - x).$$

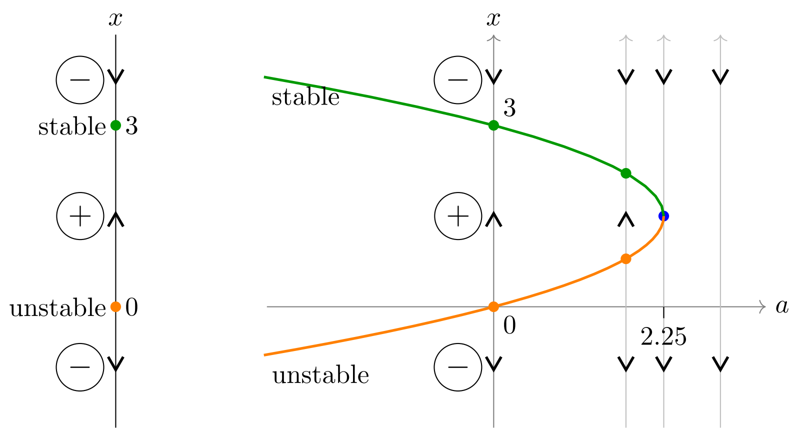
Since a is the horizontal axis, the graph of this is a sideways parabola:



Step 3. The plot divides the plane into 2 regions (inside and outside the parabola). Since the plot is the set of points where $x' = 0$, the sign of x' is the same throughout each region.

We can find those signs by testing points in each region. For example, at the point $(a, x) = (0, 1)$, we have $x' = 2 > 0$. So, inside the parabola, we have $x' > 0$. Likewise, at $(0, 4)$, $x' = -4 < 0$. So, outside the parabola, we have $x' < 0$.

Another method, which amounts to the same thing, is to use phase lines. Below, the phase line for $a = 0$ is shown on the left and also on the bifurcation diagram. On the bifurcation diagram it is the vertical line at $a = 0$.



Phaseline: $a = 0$ Bifurcation diagram

The arrows tell us the sign of x' at points on the phase line. Which, just like testing points, allows us to give the sign of x' in the two regions determined by the critical points.

These signs then tell us the stability of the critical points. In this example, the upper branch of the parabola consists of stable critical points and the lower branch consists of unstable critical points.

It is a simple matter to use the signs to add a few more phase lines to our picture. We add one through the vertex of the parabola and also ones to the left and right of the vertex.

The phase line through the vertex of the parabola shows it is semistable. The vertex is at the maximum value of a as a function of x. In this case, it's easy use calculus, or the geometry of parabolas, to find these coordinates: $a = 2.25$, $x = 1.5$.

Finally we can say when the population is sustainable: Since there is a positive stable critical point for $a < 2.25$, the population is sustainable in this region. It is not sustainable for $a \geq 2.25$.

Definition. A bifurcation point is any value of a where there is a qualitative change in the critical points.

In the previous example, the value $a = 2.25$ is the point where the critical points change –there are two critical points for $a < 2.2.5$ and none for $a > 2.25$. Therefore, $a = 2.25$ is called a bifurcation point.

Example 12.11. Suppose a population is modeled by the DE $x' = -ax + 1$, which is a constant birth-and-death rate, modified to include a constant rate of replenishment.

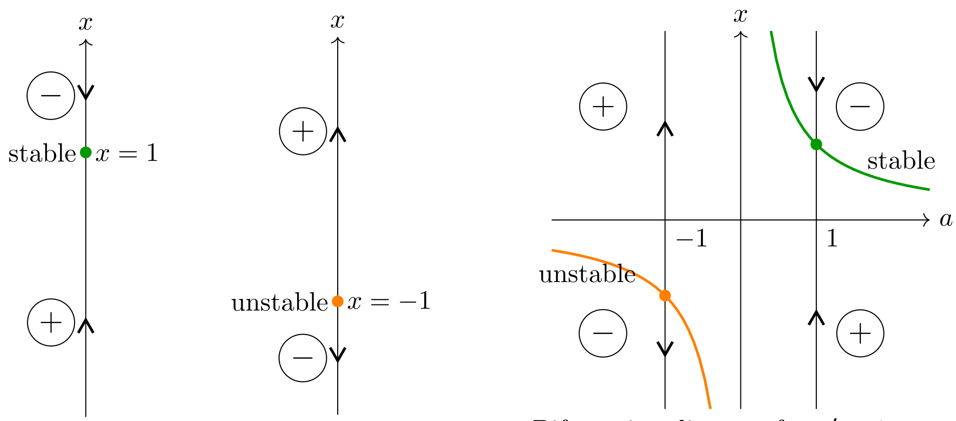
(i) Sketch the bifurcation diagram and list any bifurcation points (i.e., special values of a).

(ii) The bifurcation point(s) divide the a-axis into intervals. Illustrate one case for each interval by giving the phase line diagram. For each of these phase lines give (rough) sketches of solutions in the tx-plane.

(iii) For what values of a is the population sustainable. What happens for other values of a.

Note the MIT Mathlet The Phase Lines Mathlet https://mathlets.org/mathlets/phase-lines/ can show this system.

Solution: We answer (i) and (ii) together. The critical points are $x' = -ax + 1 = 0$. So, $x = 1/a$. We graph this in the ax-plane –it's a hyperbola with two branches. Here is the finished bifurcation diagram with two phase lines. These are explained below.



Bifurcation diagram for $x' = 1 - ax$

After plotting the critical points we see that the graph divides the ax-plane into 3 regions. In order to determine the sign of x' in each region we found phase lines for $a = 1$ and $a = -1$. These are shown at the left. Determining the direction of the arrows was straightforward and we leave it for the reader to supply the details.

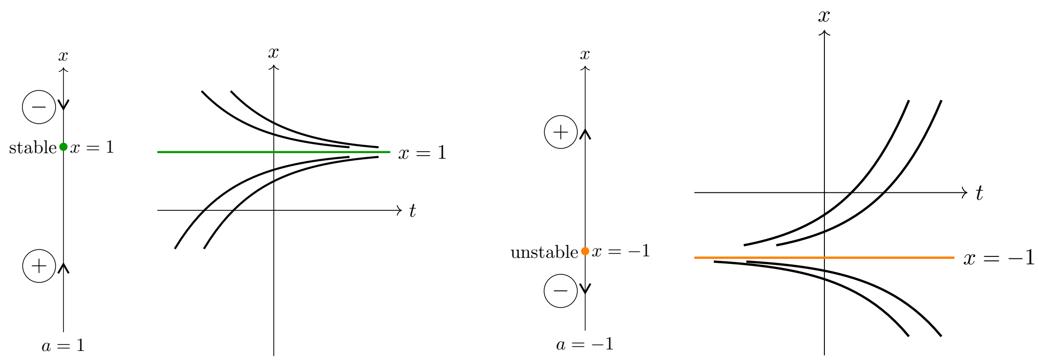
We place the phase lines on the bifurcation diagram at $a = 1$ and $a = -1$. The arrows on the phase lines then tell us the sign of x' in all 3 regions.

Once we know the sign on x', it's a simple matter to decide the stability of each part of the diagram. The stable branch is drawn in green and labeled 'stable'. Likewise the unstable branch is drawn in orange and labeled 'unstable'.

There is one bifurcation point at $a = 0$. This is a bifurcation point because the bifurcation diagram is different on either side of $a = 0$.

(iii) When $a > 0$ there is a positive stable equilibrium, so the population is sustainable. When $a \leq 0$ the population is not sustainable. In fact, it blows up to infinity.

Finally, we do our duty and sketch some solution curves based on the phase lines.



You can look at this example and the logistic with harvesting example in the Phase Lines Mathlet https://mathlets.org/mathlets/phase-lines/ phase lines applet:

12.9 Appendix: solution to logistic equation

Just for kicks, we compute the exact solution to the logistic population model

$$x' = kx(M - x)$$

This is separable. We need to use partial fractions to integrate the x side.

$$\frac{dx}{x(M-x)} = k\,dt.$$

So, $\displaystyle\int \frac{dx}{x(M-x)} = kt + C.$

Partial fractions: $\displaystyle\frac{1}{x(M-x)} = \frac{1/M}{x} + \frac{1/M}{M-x}.$

So, $\displaystyle\int \frac{dx}{x(M-x)} = \frac{\ln(|x|)}{M} - \frac{\ln|M-x|}{M} = \frac{1}{M}\ln\left(\frac{|x|}{|M-x|}\right).$

So, $\displaystyle\ln\left(\frac{|x|}{|M-x|}\right) = Mkt + C.$

Exponentiating and changing e^{MC} to C gives: $\displaystyle\frac{x}{M-x} = Ce^{Mkt}.$

Solving for x: $\displaystyle x(t) = \frac{MCe^{Mkt}}{1 + Ce^{Mkt}}.$

We can also rewrite this as $\displaystyle x(t) = \frac{MC}{e^{-Mkt} + C}.$

We were a little sloppy with the absolute values, but more care would give the same results.

Note: If $C > 0$ then the solution $x(t)$ has $0 < x < M$. If C is negative, then these solutions blow up when $e^{-Mkt} + C = 0$.
