ES.1803 Topic 15 Notes
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15 Transpose, inverse, determinant

15.1 Goals

1. Know the definition and be able to compute the inverse of any *square* matrix using row operations.

2. Know the properties of inverses. In particular, that $\det(A) \neq 0$ is equivalent to the existence of A^{-1}.

3. Know the definition and be able to compute the determinnant of any *square* matrix.

4. Know the properties of determinant.

5. Know the definition and be able to compute the transpose of any matrix.

6. Understand how elementary row operations affect the determinant and be able to use this to simplify computing determinants.

7. Know the definition of diagonal and triangular matrices and be able to easily compute their determinants and, for diagonal matrices, inverses.

8. Recall from 18.02 the method of Laplace expansion for computing inverses and determinants.

15.2 Introduction

The main point of this topic is to learn how to compute determinants (of square matrices). The main application is that the determinant is 0 exactly when the matrix has a nontrivial null space. This will be key when we learn about eigenvalues and eigenvectors.

You learned how to compute determinants in 18.02. We'll recall the methods learned there and add another method based on row reduction. This will simplify the sometimes tedious calculations. We will do something similar with inverses.

15.3 Inverses of square matrices

You saw matrix inverses in 18.02, so we will assume they are somewhat familiar.

Definition: A square matrix A has an inverse if there is another matrix, denoted A^{-1}, such that $A^{-1}A = AA^{-1} = I$.

Property. $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: It's easy to check that $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$.

Story. If you put on your sweater and then your jacket, to reverse it you have to first take off your jacket and then your sweater.

Let A be an $n \times n$ matrix. Two important questions are
Q1. Is A invertible? That is, does A have an inverse?

Q2. How do you compute A^{-1}?

We can often answer Question 1 using the following list of equivalent statements.

1. A has an inverse.
2. $\det(A) \neq 0$.
3. A has a trivial null space,i.e., the null space $= \{\mathbf{0}\}$.
4. A has rank n (we say A has full rank).
5. The columns of A are independent.
6. The echelon form of A has n pivots.
7. The reduced row echelon form of A is the identity.
8. For every choice of $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. That solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Proofs: We'll give brief arguments why numbers 2-8 follow from 1. The proof of the converses, i.e., that number 1 follows from each of 2-8 are similar. So assume that A has an inverse A^{-1}.

2. We'll see below (and you saw in 18.02) that in computing A^{-1} we divide by $\det(A)$. Since we can't divide by 0, we must have $\det(A) \neq 0$.

3. Suppose $\mathbf{v}$ is in the null space of A, so $A\mathbf{v} = \mathbf{0}$. Then, since A has an inverse, we know $\mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0}$. This shows that the only vector in the null space is $\mathbf{0}$.

4. We just showed that 1 implies A has a trivial null space. Thus it has no free variables. This implies it has n pivot variables, i.e., it has rank n.

5,6,7. These are just different ways of saying A has rank n.

8. This is obvious.

It's also worth recording these equivalences in inverse form. The following are equivalent for A

1. A does not have an inverse. (We say A is singular or non-invertible.)
2. $\det(A) = 0$.
3. A has a nontrivial null space,i.e., the null space contains non-zero vectors.
4. A has rank less than n.
5. The columns of A have some dependencies
6. The echelon form of A has fewer than n pivots.
7. The RREF of A has some all 0 rows
8. For every choice of $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has either no solutions or infinitely many solutions.

15.3.1 Matlab

Matlab makes it easy to compute the inverse of a matrix A. The function `inv(A)` returns A^{-1}. For example, to solve $A\mathbf{x} = \mathbf{b}$ in Matlab you would give the command: `x = inv(A)*b`.

15.3.2 Computing inverses.

There are a number of methods for computing the inverse of a matrix. First we remind you of the inverse of a 2 by 2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In words, swap the main diagonal elements, change the sign (without swapping) of the off diagonal elements, and divide by the determinant. You should memorize this. We will use it often and you won't want to waste time deducing it in each case.

Next, we will show how to find and inverse using elimination. A few examples will illustrate how to do this. The reason it works is straightforward, but we will relegate the explanation to an optional appendix at the end of these notes.

Example 15.1. Find the inverse of $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

Solution: We augment A by the identity and then use row reduction to bring the left-hand side to the identity.

$$\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1 \text{ and } R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 6 & 5 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - 6 \cdot R_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -7 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{R_2 = (-1/7) \cdot R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right] \xrightarrow{R_1 = R_1 - 2 \cdot R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2/7 & -5/7 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right]$$

The right half of the last augmented matrix is the inverse

$$\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/7 & -5/7 \\ -1/7 & 6/7 \end{bmatrix}$$

Example 15.2. Find the inverse of $A = \begin{bmatrix} 1 & 5 & 4 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: We augment A by the identity and use row reduction as in the previous example.

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 4 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 = R_2 - 3 \cdot R_3]{R_1 = R_1 - 4 \cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & -4 \\ 2 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - 5 \cdot R_2} \left[\begin{array}{ccc|ccc} -9 & 0 & 0 & 1 & -5 & 11 \\ 2 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 = (-1/9) \cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/9 & 5/9 & -11/9 \\ 2 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2 \cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/9 & 5/9 & -11/9 \\ 0 & 1 & 0 & 2/9 & -1/9 & -5/9 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So, $A^{-1} = \dfrac{1}{9} \begin{bmatrix} -1 & 5 & -11 \\ 2 & -1 & -5 \\ 0 & 0 & 9 \end{bmatrix}$, as you can easily verify.

Example 15.3. Let's see what happens if we try this on a matrix that doesn't have an inverse. Try to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\;\;\;\;R_3 = R_3 - 7 \cdot R_1\;\;\;\;]{R_2 = R_2 - 4 \cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 2 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

We've reached an impasse. The matrix A only has 2 pivots, so it cannot be row reduced to the identity, i.e., it has no inverse.

Question: What is the $\det(A)$?

15.3.3   Diagonal and triangular matrices

It is simple to find the inverse of a diagonal matrix. Here are some examples.

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \\ 0 & 0 & c^{-1} & 0 \\ 0 & 0 & 0 & d^{-1} \end{bmatrix}$$

Triangular matrices require more work, but at least we only have to do elimination in one direction.

Example 15.4. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

Solution: We augment and row reduce from the top down:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\;\;R_3 = R_3 - 4 \cdot R_1\;\;]{R_2 = R_2 - 2 \cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 5 & 6 & -4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = (1/3) \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 5 & 6 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 5 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 6 & -2/3 & -5/3 & 1 \end{array}\right] \xrightarrow{R_3 = (1/6) \cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/9 & -5/18 & 1/6 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1/9 & -5/18 & 1/6 \end{bmatrix}.$

15.3.4   Laplace expansion using cofactors

In 18.02 you learned how to find the inverse using cofactors (also called the adjoint method). For completeness, we review this method in the appendix at the end of these notes. Unless we specify a method for finding an inverse, e.g., by row reduction, you may use any method you want, including Laplace expansion.

15.4   Determinants

We can take the determinant of a square matrix A. We will write $\det(A)$ or $|A|$ for the determinant of A. Since this is part of 18.02, we will assume you have seen determinants before. For us, the most important use of determinants is to check if a matrix has a trivial null space. These were Properties 2 and 3 in Section 15.3:

$$\text{Null}(A) \text{ is trivial if and only if } \det(A) \neq 0.$$
$$\text{Null}(A) \text{ is nontrivial if and only if } \det(A) = 0.$$

Properties of determinants:
1. The determinant is linear in each column and linear in each row.
2. $\det(I) = 1$.
3. Swapping rows changes the sign of the determinant.
4. Scaling a row scales the determinant.
5. Adding a multiple of one row to another doesn't change the determinant.
6. $\det(AB) = \det(A)\det(B)$.

15.4.1   Laplace expansion using minors

In 18.02 you learned how to find the determinant using minors. We give a review of that method in the appendix at the end of the notes for this topic. Unless we specify a method for finding the determinant, e.g., by row reduction, you may use any method you want including Laplace expansion.

15.4.2   The 2 by 2 case

You should know the determinant of a 2×2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

15.4.3   Easy determinants

The easiest determinants to compute are for diagonal and triangular matrices. In these cases the determinant is just the product of the diagonal entries.

$$\textbf{Diagonal: } \det \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} = abcd.$$

$$\textbf{Upper triangular: } \det \begin{bmatrix} a & e & f & g \\ 0 & b & h & i \\ 0 & 0 & c & j \\ 0 & 0 & 0 & d \end{bmatrix} = abcd$$

$$\textbf{Lower triangular: } \det \begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{bmatrix} = acfj.$$

Identical rows: If A has two identical rows then $\det A = 0$.

Proof: Swapping the rows leaves A and therefore, $\det(A)$ unchanged. But (property 3), it also changes the sign of the determinant. Only 0 stays the same when you change sign. Therefore, $\det(A) = 0$.

15.4.4   Matlab

Matlab makes it easy to compute the determinant of a matrix A. The function `det(A)` returns $\det(A)$.

15.4.5   Finding the determinant using row reduction

Since we know how the elementary row operations affect the determinant we can use row reduction to compute the determinant of a matrix. We'll illustrate with an example.

Example 15.5. Find the determinant of $A = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

Solution: We use row reduction until A is in triangular form. At each step we keep track of the effect on the determinant.

$$\begin{bmatrix} 0 & 4 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{swap rows; } \det \times (-1)} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3 \cdot R_1; \text{ det unchanged}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 1 \\ 0 & -5 & -4 \end{bmatrix}$$

$$\xrightarrow{R_2 = (1/4) \cdot R_2; \det \times (1/4)} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1/4 \\ 0 & -5 & -4 \end{bmatrix} \xrightarrow{R_3 = R_3 + 5 \cdot R_2; \text{ det unchanged}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1/4 \\ 0 & 0 & -11/4 \end{bmatrix}$$

The last matrix is triangular, so its determinant is the product of its diagonal entries, i.e., $-11/4$. Following the changes in the determinant caused by the row operations, we have

$$(-1) \cdot \left(\frac{1}{4}\right) \det(A) = \det \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1/4 \\ 0 & 0 & -11/4 \end{bmatrix} = \frac{-11}{4} \Rightarrow \det(A) = 11.$$

15.5   Transpose

For us, the transpose will be a convenient tool for calculation and presenting matrices. For example, in Matlab we can use the transpose and matrix multiplication to compute dot (inner) products. There is a lot more to transposes than we will see. You should take 18.06 to learn more.

To take the transpose of a matrix you change rows into columns. We'll use the notation A^T for the transpose of A.

Example 15.6. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Note. Transpose turns a 3×2 matrix into a 2×3 matrix. In general, it turns an $n \times m$ matrix into an $m \times n$ matrix.

In terms of entries, the i, j entry of A^T equals the j, i entry of A. In symbols: $(A^T)_{i,j} = A_{j,i}$.

You can check that the dimensions make sense: If A is $m \times n$ and B is $n \times p$ then AB is $m \times p$, so $(AB)^T$ is $p \times m$. Likewise, we can show that $B^T A^T$ is $p \times m$.

Of course we can prove this property, but in 18.03 we are not particularly concerned with the proof, so, for anyone who is interested, we'll put it in the appendix at the end of the notes for this topic.

Symmetric matrices.   A square matrix A is symmetric if $A = A^T$.

Example 15.7. The following matrices are symmetric

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \quad \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Notes: **1.** Symmetric means symmetric around the main diagonal.
2. Diagonal matrices are always symmetric.
3. It doesn't make sense to ask if a non-square matrix is symmetric.
4. Matlab uses a prime to mean transpose, e.g., [1,2; 3,4; 5,6]$'$.

Symmetric matrices are an extremely important class of matrices, which arise in many applications. Unfortunately, we won't have time to do much with them in 18.03.

15.5.1 Inner products and transposes

In 18.02 you learned about the dot product of two vectors, e.g.,

$$(1, 2, 3) \cdot (2, -1, 4) = 2 - 2 + 12 = 12.$$

Since the dot is also used for multiplication, we are going to (mostly) quit using the dot notation and also rename the dot product as the inner product. Here is our new terminology and notation.

Definition: The inner product of two vectors $\mathbf{v}$ and $\mathbf{w}$ is denoted $\langle \mathbf{v}, \mathbf{w} \rangle$. If $\mathbf{v}$ and $\mathbf{w}$ are column vectors in $\mathbf{R}^3$ then

$$\langle \mathbf{v}, \mathbf{w} \rangle = \left\langle \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

This is not restricted to vectors in $\mathbf{R}^3$, we can define the inner product between vectors in $\mathbf{R}^n$ for any n.

The inner product of two column vectors can be computed as a matrix multiplication using the transpose.

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{w} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

We will not do very much with inner product for now, though it will come up again later. For now, the most important thing to remember is that two vectors are orthogonal if their inner product is 0.

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0 \Leftrightarrow \mathbf{v} \text{ and } \mathbf{w} \text{ are orthogonal.}$$

15.5.2 Saving space

Now that we have the transpose, we can use it to save space on the page. Instead of always writing column vectors vertically, we can use the transpose to write them horizontally, e.g.,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$$

15.6 Appendix

This appendix contains some review and some more technical material. The technical material is just for your reading pleasure. You will not be asked to reproduce it for ES.1803.

15.6.1   Review: Determinants by Laplace expansion along a row or column

The i, j minor of a matrix is the determinant after removing the ith row and jth column, i.e., the row and column intersecting at the i, j entry.

Example 15.8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find all the minors that go with the second column.

Solution: The second column has the (1,2), (2,2) and (3,2) entries. The (1,2) minor is the 2×2 determinant of the matrix after crossing out the row and columns through the (1,2) entry

$$\text{(1,2) minor} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6$$

Likewise, for the other two entries in the second column.

$$\text{(2,2) minor} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = -12, \qquad \text{(3,2) minor} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -6$$

We can compute the determinant of the $n \times n$ matrix A by expanding along any row or column.

$$\det A = \text{sum along the row of (checkerboard sign)} \cdot \text{(entry)} \cdot \text{(minor)}$$

As a formula, expanding along the ith row we have

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot (i, j) \text{ minor.}$$

To expand along a column, you fix j and sum over i.

Example 15.9. Expand $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ along the middle column.

Solution: (Long, drawn out version.) First we draw a line through the second column:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now we use the sign checkerboard $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ to compute the determinant using the entries and minors along the second column.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -2 \overbrace{\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}^{(1,2)\text{minor}} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -2(-6) + 5(-12) - 8(-6) = 0$$

(1,2) matrix entry

(1,2) checkerboard sign

Here, the sign in front of each term comes from the checkerboard, e.g., the (1,2) checkerboard entry is a minus sign, so that term gets a minus sign.

The same process works expanding along any row or column.

Example 15.10. Compute $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ along the top row

Solution: (Short, concise version.) Determinant $= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 0$

Example 15.11. Compute $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 0 & 7 \\ 8 & 0 & 9 \end{vmatrix}$

Solution: Use second column: $\det = -2 \cdot \begin{vmatrix} 5 & 7 \\ 8 & 9 \end{vmatrix} + 0 \cdot * - 0 \cdot * = 22$. (To save time, we didn't bother computing the minors that were multiplied by 0.)

15.6.2 Using row reduction on the augmented matrix to find the inverse

Here we will explain why this technique works. The key fact here is that every elementary row operation corresponds to multiplication by a matrix on the left. We illustrate by row reducing our favorite matrix to the identity.

$$\begin{aligned}
&\text{Original matrix:} &&A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \\
&\text{Swapping } R_1 \text{ and } R_3: &&\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \\
&R_2 = R_2 - 6R_1: &&\begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} \\
&R_2 = (-1/7) \cdot R_2: &&\begin{bmatrix} 1 & 0 \\ 0 & -1/7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\
&R_1 = R_1 - 2 \cdot R_2 &&\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

If you put all the matrix multiplications together we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

That is

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

This product is exactly what we get by applying the same sequence of elementary row operations to the identity matrix on the right side of the augmented matrix $(A|I)$.

15.6.3   Finding inverses using cofactors (the Laplace or adjoint method)

We have a simple formula for finding the inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For bigger (square) matrices finding inverses is more involved. One algorithm for doing this is called the adjoint or Laplace method.

The **step-by-step algorithm** is the following:

1. Start with A.

2. Find the matrix of minors.

3. Find the matrix of cofactors.

4. Find the adjoint.

5. Divide by $\det(A)$.

Of course, we have to explain what each of these things is. We will over the next four examples, explaining one item at a time.

For these examples let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$.

Matrix of minors. We definied the i, j minor of a matrix in Section 15.6.1. The matrix of minors of A is just the matrix made up of all the minors. The i, j-entry of the matrix of minors is the i, j-minor of A.

Example 15.12. Find the matrix of minors of A.

Solution: A is a 3×3 matrix so its matrix of minors is also 3×3. Here is the computation for each minor:

$$1, 1 \text{ minor:} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}; \quad 1, 1\text{-minor} = \begin{vmatrix} 5 & 6 \\ 2 & 0 \end{vmatrix} = -12; \quad \text{matrix of minors} = \begin{bmatrix} -12 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$$1, 2 \text{ minor:} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}; \quad 1, 2\text{-minor} = \begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} = -6; \quad \text{matrix of minors} = \begin{bmatrix} -12 & -6 & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$$1, 3 \text{ minor:} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}; \quad 1, 3\text{-minor} = \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 3; \quad \text{matrix of minors} = \begin{bmatrix} -12 & -6 & 3 \\ * & * & * \\ * & * & * \end{bmatrix}$$

$2, 1$ minor: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$; $2, 1\text{-minor} = \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} = -6;$ matrix of minors $= \begin{bmatrix} -12 & -6 & 3 \\ -6 & * & * \\ * & * & * \end{bmatrix}$

There are 5 more minors to compute. We show each of them, but without labels. You should practice by naming them and computing their value.

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$;

The entire matrix of minors is therefore: $\begin{bmatrix} -12 & -6 & 3 \\ -6 & -3 & 0 \\ -3 & -6 & -3 \end{bmatrix}$.

Matrix of cofactors. Recall the checkerboard of signs we used for computing the determinant: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$. To compute the matrix of cofactors of A, you change the signs in the matrix of minors according to the checkerboard.

Example 15.13. Find the matrix of cofactors for the matrix A in the previous example.

Solution: The matrix of minors is $\begin{bmatrix} -12 & -6 & 3 \\ -6 & -3 & 0 \\ -3 & -6 & -3 \end{bmatrix}$ So the matrix of cofactors is $\begin{bmatrix} -12 & 6 & 3 \\ 6 & -3 & 0 \\ -3 & 6 & -3 \end{bmatrix}$.

(Look carefully at how we changed signs to go from minors to cofactors.)

Adjoint. To make the adjoint matrix you take the transpose of the cofactors matrix, i.e., switch the rows and columns of the cofactors matrix.

Example 15.14. Find the adjoint matrix for the matrix A in the previous examples.

Solution: The matrix of cofactors is $\begin{bmatrix} -12 & 6 & 3 \\ 6 & -3 & 0 \\ -3 & 6 & -3 \end{bmatrix}$. So the adjoint is $\begin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & 6 \\ 3 & 0 & -3 \end{bmatrix}$.

Example 15.15. Find the inverse for the matrix A in the previous examples.

Solution: We can find $\det A = 9$ using the minors for the first row computed in Example 15.12. The matrix of cofactors is $\begin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & 6 \\ 3 & 0 & -3 \end{bmatrix}$, so the inverse is

$$A^{-1} = \frac{1}{9} \begin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & 6 \\ 3 & 0 & -3 \end{bmatrix}.$$

Finding the inverse. Divide the matrix of cofactors by $\det A$.

The next example will show a good way to organize the computation.

Example 15.16. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

Solution: In order to have an inverse we need $\det(A) \neq 0$. So our first step is to compute the determinant. We do this by expansion along the first row:

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 1(-4) - 2(-2) + 3(3) = 9.$$

Since $\det(A) \neq 0$, the inverse exists and we can proceed with the algorithm to compute A^{-1}. Only the first step requires any real computation.

The algorithm says to first compute the matrix of minors. Notice that we found the minors for the first row when we computed the determinant. We can reuse those and only need to compute the other 6. (Actually we'll just use the answers from the previous examples.)

1. Matrix of minors $= \begin{bmatrix} -4 & -2 & 3 \\ -6 & -3 & 0 \\ 1 & -4 & -3 \end{bmatrix}$ (compute each minor).

2. Matrix of cofactors $= \begin{bmatrix} -4 & 2 & 3 \\ 6 & -3 & 0 \\ 1 & 4 & -3 \end{bmatrix}$ (apply checkerboard).

3. Adjoint $= \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & 4 \\ 3 & 0 & -3 \end{bmatrix}$ (swap rows and columns).

4. Inverse: $A^{-1} = \frac{1}{9} \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & 4 \\ 3 & 0 & -3 \end{bmatrix}$ (divide by $\det(A)$).

We can check this by multiplying by multiplying $A^{-1} \cdot A$ and seeing that we get I. (You'll have to do the actual computation.)

$$A^{-1} \cdot A = \frac{1}{9} \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & 4 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = I.$$

Fun note: This algorithm works for the 2×2 case as well. You should try it out, it's very fast.

15.6.4 Proof that $(AB)^T = B^T A^T$

We'll use the following notation involving indices. Let the (i, j) entry of A be $A_{i,j}$. By the definition of transpose we have $(A^T)_{j,i} = A_{i,j}$. Likewise for other matrices. In order to show that $(AB)^T = B^T A^T$ we have to show that $((AB)^T)_{k,i} = (B^T A^T)_{k,i}$. We do this by

keeping track of indices while multiplying matrices. Since $(AB)_{i,k} = \sum_j A_{i,j} B_{j,k}$ we have

$$((AB)^T)_{k,i} = (AB)_{i,k} = \sum_j A_{i,j} B_{j,k} = \sum_j B_{j,k} A_{i,j} = \sum_j (B^T)_{k,j} (A^T)_{j,i} = (B^T A^T)_{k,i} \qquad \text{QED}$$

15.6.5   Left and right inverses

A left inverse for a matrix A is a matrix L such that left-multiplication by L gives the identity, e.g.,

$$L \cdot A = I$$

The definition of a right inverse is similar.

Non-square matrices can have one-sided inverses. For example the matrix $A = \begin{bmatrix} 6 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ has a right inverse (in fact many of them). For example,

$$\begin{bmatrix} 6 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2/7 & -5/7 \\ -1/7 & 6/7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But A has no left inverse. Likewise, there are matrices with left inverses but no right inverses.

Here are some facts about these one-sided inverses. We won't give details, but you do have all the tools to understand the details: ask if you're interested.

1. A square matrix either has a single two-sided inverse, i.e., both a left and a right inverse, or it has no inverses of any kind.

2. If $n < m$ then an $n \times m$ matrix A cannot have a left inverse. If the rank of A is n then it has a right inverse. The example just above, illustrates this for A a 2×3 matrix of rank 2.

3. If $n > m$ then an $n \times m$ matrix A cannot have a right inverse. If the rank of A is m then it has a left inverse.

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