ES.1803 Topic 17 Notes

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17 Matrix methods for solving systems of DEs

17.1 Goals

1. Be able to solve constant coefficient linear systems using eigenvalues and eigenvectors. Do this when there are real or complex eigenvalues.
2. Understand and appreciate the abstraction of matrix notation.
3. Be able to convert a higher order linear DE equation into a *companion system* of coupled first-order equations.
4. See some physical settings modeled by systems of equations.

17.2 Introduction

In this topic we will look in detail at solving linear constant coefficient systems of differential equations using eigenvalues and eigenvectors. We will need to consider cases of real, complex and repeated eigenvalues. (We will only touch on the case of repeated eigenvalues.).

An important idea is that any higher order differential equation can be converted into a system of first-order equations. This means that our old friend $P(D)x = 0$ can be converted into a system and solved with these methods. This is useful because it is more natural to formulate numerical algorithms for first-order systems than for higher order equations. This is partly explained by the first section below, which looks at the utility of matrix notation.

17.3 Matrix notation and why we like it

We have been using matrix notation for algebraic systems and systems of differential equations. Let's remind ourselves why it's helpful in organizing our thinking.

One of the simplest algebraic equations is

$$ax = b, \quad \text{where } a \text{ and } b \text{ are constants and } x \text{ is the unknown.} \tag{1}$$

We easily solve this for x: $x = a^{-1}b$ (provided $a \neq 0$).

On the face of it a system of algebraic equations seem more complicated. For example consider the following system of two equations in two unknowns:

$$\begin{aligned} 6x + 5y &= 2 \\ x + 2y &= 3 \end{aligned}$$

We could solve this by elimination, but here our interest in writing this out abstractly. In matrix form the system and its solution become

$$\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

If we give names: $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then the system and its solution become

$$A\mathbf{x} = \mathbf{b} \;\Rightarrow\; \mathbf{x} = A^{-1}\mathbf{b}.$$

At this level of abstraction we see that the system and its solution are just like those of our simplest equation. (One small difference is that we need to take more care with the order of matrix multiplication than we do with scalar multiplication.)

For differential equations our simplest and favorite equation is

$$x' = ax.$$

Written in matrix form, a linear system of DEs looks similar.

Example 17.1. As above, let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Write the following system in a form that resembles our favorite DE.

$$\begin{aligned} x' &= 6x + 5y \\ y' &= x + 2y \end{aligned}$$

Solution: In matrix form this becomes

$$\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad \text{or} \qquad \mathbf{x}' = A\mathbf{x}.$$

The right hand equation looks just like our favorite DE.

Note: we will call A the coefficient matrix of the system.

17.4 Solving homogeneous DEs using matrix methods

17.4.1 Review

In the previous topic we looked briefly at solving linear, homogeneous, constant coefficient systems using matrix methods. Recall that we used the method of optimism to guess a solution of the form $e^{\lambda t}\mathbf{v}$. Substituting this in the equation leads immediately to the fact that λ must be an eigenvalue and $\mathbf{v}$ an eigenvector.

We'll review the process with brief explanations. Later, we will write model solutions that skip directly to the characteristic equation.

Example 17.2. Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

This is a linear, homogeneous, constant coefficient system of DEs.

Solution: Try $\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t}\mathbf{v}$.

Substitution gives: $\lambda e^{\lambda t}\mathbf{v} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} e^{\lambda t}\,\mathbf{v} \Leftrightarrow \boxed{\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \mathbf{v} = \lambda\,\mathbf{v}.}$

The boxed equation is the eigenvector/eigenvalue equation, where λ is the eigenvalue and $\mathbf{v}$ is the corresponding eigenvector.

We know how to find eigenvalues and eigenvectors:

Characteristic equation: $\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda = 4, 1.$

Eigenvectors are in $\text{Null}(A - \lambda I)$:

$\lambda_1 = 4$: $\quad A - \lambda I = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$. Basic eigenvector: $\quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda_2 = 1$: $\quad A - \lambda I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. Basic eigenvector: $\quad \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Two *modal* solutions are $\mathbf{x_1}(t) = e^{4t}\mathbf{v_1} = e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x_2}(t) = e^{t}\mathbf{v_2} = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The general solution is $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1 e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Note: Each of the solutions $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is called a normal mode or modal solution.

17.4.2 Complex eigenvalues

We handle complex eigenvalues in exactly the same manner as we did complex characteristic roots for ordinary differential equations.

Theorem: Suppose A is a real matrix. Consider the DE: $\mathbf{x}' = A\mathbf{x}$.

If $\mathbf{z}$ is a complex solution to this DE then both the real and imaginary parts of $\mathbf{z}$ are also solutions.

Proof: Suppose $\mathbf{z} = \mathbf{x}_1 + i\mathbf{x}_2$ then

$$\begin{aligned} & \mathbf{z}' = A\mathbf{z} \\ \Leftrightarrow & (\mathbf{x}_1 + i\mathbf{x}_2)' = A(\mathbf{x_1} + i\mathbf{x}_2) \\ \Leftrightarrow & \mathbf{x}_1' + i\mathbf{x}_2' = A\mathbf{x_1} + iA\mathbf{x_2} \end{aligned}$$

If two complex numbers are equal then their real parts must be equal and so must the imaginary parts. Therefore, the equation above shows

$$\mathbf{x}_1' = A\mathbf{x}_1 \quad \text{and} \quad \mathbf{x}_2' = A\mathbf{x}_2.$$

That is, $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions to the DE.

Notes:

1. The proof is just linearity written out the long way.

2. To be perfectly careful we should say that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the real and imaginary parts of $\mathbf{z}$, but this is clear from the context.

The next example illustrates the use of this theorem.

Example 17.3. Find the general, real-valued solution to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Solution: Characteristic equation: $|A - \lambda I| = \begin{vmatrix} 3-\lambda & -5 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13 = 0$

Solving, we get $\lambda = 2 \pm 3i$. (Complex roots always come in conjugate pairs.)

Eigenvectors: Find a basis for Null$(A - \lambda I)$.

$\lambda = 2 + 3i$: $(A - \lambda I) = \begin{bmatrix} 1-3i & -5 \\ 2 & -1-3i \end{bmatrix}$.

By inspection, a basic eigenvector is $\mathbf{v_1} = \begin{bmatrix} 5 \\ 1-3i \end{bmatrix}$.

Note: There is no need to compute the second eigenvector since it is just the complex conjugate of the first one.

This gives us a complex-valued solution

$$\begin{aligned} \mathbf{z_1}(t) &= e^{(2+3i)t} \begin{bmatrix} 5 \\ 1-3i \end{bmatrix} = e^{2t}(\cos 3t + i\,\sin 3t) \begin{bmatrix} 5 \\ 1-3i \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 5\cos 3t + i5\sin 3t \\ \cos 3t + 3\sin 3t + i(-3\cos 3t + \sin 3t) \end{bmatrix} \end{aligned}$$

Just for completeness we give its complex conjugate which is also a solution

$$\mathbf{z_2}(t) = \overline{\mathbf{z}_1(t)} = e^{(2-3i)t} \begin{bmatrix} 5 \\ 1+3i \end{bmatrix} = e^{2t} \begin{bmatrix} 5\cos 3t - i5\sin 3t \\ \cos 3t + 3\sin 3t - i(-3\cos 3t + \sin 3t) \end{bmatrix}$$

The theorem above tells us that The real and imaginary parts of $\mathbf{z_1}$ are both solutions:

$$\begin{aligned} \mathbf{x_1}(t) &= e^{2t} \begin{bmatrix} 5\cos 3t \\ \cos 3t + 3\sin 3t \end{bmatrix} \\ \mathbf{x_2}(t) &= e^{2t} \begin{bmatrix} 5\sin 3t \\ -3\cos 3t + \sin 3t \end{bmatrix}. \end{aligned}$$

As always, the general, *real-valued* solution is given by superposition

$$\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1 e^{2t} \begin{bmatrix} 5\cos 3t \\ \cos 3t + 3\sin 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 5\sin 3t \\ -3\cos 3t + \sin 3t \end{bmatrix}.$$

17.4.3 Repeated roots (2 by 2 case only)

Repeated eigenvalues complicate matters somewhat. We will study this by looking at two examples.

Example 17.4. (**Complete case**) Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Solution: This is a diagonal matrix so the eigenvalues are $\lambda = 5, 5$.

For $\lambda = 5$ the matrix $A - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The null space of this matrix is all of $\mathbf{R}^2$. That is, every vector is an eigenvector i.e., the eigenspace is 2 dimensional. Since we only need to choose two independent eigenvectors, we can choose the standard basis vectors:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(Any other independent pair would work as well.)

Thus the general solution to the DE is $\mathbf{x} = c_1 e^{5t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{5t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

This is called the complete case because we have a full complement of basic solutions. That is, we have two independent solutions to our second-order system.

The next example looks at the so-called defective case. The name comes from the following ideas. If a matrix has a repeated eigenvalue we would like an independent eigenvector for each time the eigenvalue is repeated. The matrix is defective if this is not the case.

Example 17.5. (**Defective case**) Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Solution: First we find the eigenvalues: The characteristic equation is

$$|A - \lambda\, I| = \lambda^2 - 10\lambda + 25 = 0.$$

So the eigenvalues are repeated: $\lambda = 5,\ 5$.

Next we find the basic eigenvectors $\mathbf{v}$. As usual, we need find to a basis of Null$(A - \lambda\, I)$.

For $\lambda = 5$: $\quad A - \lambda I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$.

The row reduced echelon form (RREF) of the coefficient matrix is $R = \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$.

This has only one free variable, so the eigenspace is only one dimensional. A basis is given by $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

This eigenvector gives us one solution to the DE: $\mathbf{x_1} = e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

As we said, this case is defective. The system is second-order but the eigenmethods only found one solution. We'll use a magic algorithm to find a second solution. Below we'll see why the magic worked. You will need to take 18.06 (or even better 18.701) for more insight on why this works.

The first step of the algorithm is to solve $(A - \lambda I)\mathbf{v_2} = \mathbf{v_1}$. That is,

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Using row reduction (or by inspection) we find that one solution is $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The algorithm now tells us that a second solution to the DE is

$$\mathbf{x_2} = te^{5t}\mathbf{v_1} + e^{5t}\mathbf{v_2} = te^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now that we have two solutions we can give the general solution to the DE:

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x_1} + c_2\mathbf{x_2} \\ &= c_1\, e^{5t}\begin{bmatrix}1\\2\end{bmatrix} + c_2\left(te^{5t}\begin{bmatrix}1\\2\end{bmatrix} + e^{5t}\begin{bmatrix}1\\1\end{bmatrix}\right)\end{aligned}$$

Abstract version of defective case

The example above is complicated by actual computations. Here is the abstract version of the algorithm for the defective case. We check that the result is a solution by plugging it into the DE.

The algorithm uses two vectors
1. An eigenvector $\mathbf{v_1}$, i.e., $A\mathbf{v_1} = \lambda\mathbf{v_1}$

2. A vector $\mathbf{v_2}$ that satisfies $(A - \lambda\, I)\mathbf{v_2} = \mathbf{v_1}$.

$\mathbf{v_2}$ is called a generalized eigenvector. In the proof below, we will need to use this in the form: $A\mathbf{v_2} = \mathbf{v_1} + \lambda\mathbf{v_2}$.

We assert that $\mathbf{x_1}(t) = e^{\lambda t}\mathbf{v_1}$ and $\mathbf{x_2}(t) = te^{\lambda t}\mathbf{v_1} + e^{\lambda t}\mathbf{v_2}$ are independent solutions to the DE.

Proof: We know that x_1 is the eigenvector solution. To check that $\mathbf{x_2}$ is a solution, we plug it into the DE and check that both sides of the equation are the same.

$$\begin{aligned}\text{(left side) } \mathbf{x_2}' &= \lambda te^{\lambda t}\mathbf{v_1} + e^{\lambda t}\mathbf{v_1} + \lambda e^{\lambda t}\mathbf{v_2} = \lambda te^{\lambda t}\mathbf{v_1} + e^{\lambda t}(\mathbf{v_1} + \lambda\mathbf{v_2}) \\ \text{(right side)} A\mathbf{x_2} &= te^{\lambda t}A\mathbf{v_1} + e^{\lambda t}A\mathbf{v_2} = \lambda te^{\lambda t}\mathbf{v_1} + e^{\lambda t}(\mathbf{v_1} + \lambda\mathbf{v_2})\end{aligned}$$

Comparing both sides we see that $\mathbf{x_2}' = A\mathbf{x_2}$. That is, $\mathbf{x_2}$ is a solution.

17.5 Companion systems

Early in 18.03 we learned how to solve ordinary differential equations $P(D)x = 0$. For example $x'' + 8x' + 7x = 0$. In this section we will convert a higher order ordinary differential equation to a system of first-order equations.

Example 17.6. Convert the ODE $x'' + 8x' + 7x = 0$ to a system of first-order equations.

Solution: Introduce a second variable $y = x'$. Our ODE then becomes

$$y' + 8y + 7x = 0.$$

Writing out the equations for x' and y' we get

$$\begin{aligned} x' &= \quad\quad\;\; y \\ y' &= -7x - 8y \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0 & 1\\-7 & -8\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

The system is called the **companion system** to the original ODE. We call the coefficient matrix the **companion matrix**.

We will sometimes refer to the method of converting an ODE to a system as **anti-elimination**. This is because elimination is a process of removing variables and equations, so anti-elimination is a process of adding variables and equations.

Example 17.7. Find the companion system for the ODE $x''' + 2x'' + 5x' + 7x = 0$.

Solution: Let $y = x'$ and $z = y' = x''$. The ODE becomes $z' + 2z + 5y + 7x = 0$. So our companion system is

$$\begin{array}{rcrcrcr} x' & = & & & y & & \\ y' & = & & & & & z \\ z' & = & -7x & - & 5y & - & 2z \end{array} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

17.6 Physical examples

Example 17.8. Population models
Suppose we have two countries with time varying populations x and y. Suppose also that the natural growth rate in the countries is 2% and 2% respectively. In addition every year 3% of the country 1 moves to country 2 and 1% of country 2 moves to country 1.

Give a system of differential equations modeling this scenario. Assume initial populations of $x(0) = 2$ and $y(0) = 2$ (in units of one million). Solve the system and interpret the eigenvectors in terms of populations.

Solution: We have

$$\begin{array}{ll} x' & = 0.02x - 0.03x + 0.01y \quad = -0.01x + 0.01y \\ y' & = 0.03x + 0.02y - 0.01y \quad = 0.03x + 0.01y \end{array} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0.01 & 0.01 \\ 0.03 & 0.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We solve by finding eigenvalues and eigenvectors.

Characteristic equation: $\begin{vmatrix} -0.01 - \lambda & 0.01 \\ 0.03 & 0.01 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0.02, -0.02$

Eigenvectors (basis of $\text{Null}(A - \lambda I)$, where A is the coefficient matrix:

$\lambda_1 = 0.02$: $A - \lambda I = \begin{bmatrix} -0.03 & 0.01 \\ 0.03 & -0.01 \end{bmatrix}$. Basic eigenvector: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = -0.02$: $A - \lambda I = \begin{bmatrix} 0.01 & 0.01 \\ 0.03 & 0.03 \end{bmatrix}$. Basic eigenvector: $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The general solution is

$$\begin{bmatrix} x(t) \\ y(y) \end{bmatrix} = c_1 e^{0.02\,t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-0.02\,t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

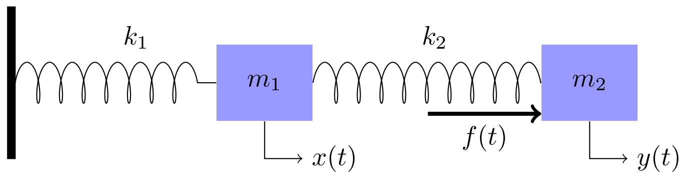
The initial conditions produce $c_1 = 1$ and $c_2 = 1$. So

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{0.02\,t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + e^{-0.02\,t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Over time the $e^{-0.02t}$ term will go to 0 and the populations will grow exponentially and in a ratio of $x/y \approx 1/3$.

Some eigenvectors may have negative entries and some eigenvalues may be negative or complex. However, any population vector is a combination of these pure modes.

Example 17.9. Coupled springs. Suppose we have two masses and springs configured as shown.



x is the displacement of m_1 from its equilibrium position.

y is the displacement of m_2 from its equilibrium position.

(So the amount that Spring 2 is stretched is $y - x$.)

$f(t)$ is a time-varying force applied to m_2.

Using Hooke's law, we get the following system of equations

$$\begin{aligned} m_1\ddot{x} &= -k_1 x + k_2(y - x) \\ m_2\ddot{y} &= -k_2(y - x) + f(t) \end{aligned}$$

We can rearrange this to be

$$\begin{aligned} \ddot{x} &= -\frac{k_1 + k_2}{m_1}x + \frac{k_2}{m_1}y \\ \ddot{y} &= \frac{k_2}{m_2}x - \frac{k_2}{m_2}y + \frac{f(t)}{m_2} \end{aligned}$$

The system is fourth-order because it consists of 2 second-order equations. You should think about how you would produce a companion system of 4 first-order equations.

This system is illustrated by the applet https://mathlets.org/mathlets/coupled-oscillators/ (You'll have to set one of the spring constants to 0.)

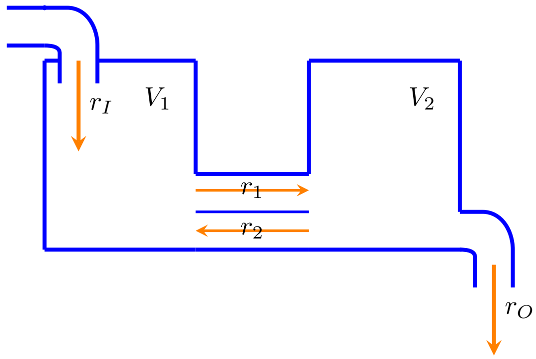
Example 17.10. Salt tanks. Suppose we have two tanks containing a salt solution. Initially the volume of water in the tanks is V_1 and V_2 respectively. Pure water flows into Tank 1 from the outside at r_I liters/minute. Solution flows out of Tank 2 at a rate of r_O liters/min. Solution is exchanged between the tanks, as shown, at the rates r_1 and r_2 in liters/min.

Suppose the rates and volumes are:

$r_I = 20$ (pure water), $r_1 = 10$, $r_2 = 30$, $r_O = 20$

$V_1 = 100$ liters, $V_2 = 200$ liters.

Note that the flow rates are balanced, so that V_1 and V_2 do not change.



Write a system of DEs modeling the amount of salt in each tank.

Solution: Let x be the grams of salt in Tank 1 and let y be the grams of salt in Tank 2.

Before starting, let's note that because pure water is being added all the salt will eventually be flushed out of the tanks, i.e., both x and $y \to 0$ in the long run. We should check that our answer reflects this.

Now for the model: x' = rate salt into Tank 1 - rate salt out of Tank 1).

rate in = flow · concentration = $r_2 \cdot \frac{y}{V_2} = 10$ l/min · y g/200 l $= \frac{10}{200} y$ g/min.

rate out $= r_1 \cdot \frac{x}{V_1} = \frac{30}{100} x$ g/min.

Thus, $\quad x' = -\dfrac{3}{10} x + \dfrac{1}{20} y$

Likewise for y': rate in $= r_1 \cdot \frac{x_2}{V_2}$, rate out $= (r_2 + r_O) \cdot \frac{y}{V_2}$

So $y' = \frac{3}{10} x - \frac{3}{20} y$.