ES.1803 Topic 18 Notes
Jeremy Orloff

18 Matrix exponential, exponential and sinusoidal input

This topic is no longer on the syllabus. We post these notes for anyone who is interested. Since we have already covered inhomogeneous, constant coefficient, linear DEs and homogeneous systems, linear systems with input is not a big step.

18.1 Goals

1. Know the definition of the matrix exponential.

2. Be able to compute the matrix exponential from eigenvalues and eigenvectors.

3. Be able to use the matrix exponential to solve an IVP for a constant coefficient linear system of differential equations.

4. Be able to derive and apply the exponential response formula for constant coefficient linear systems with exponential input.

5. Be able to solve linear constant coefficient systems with sinusoidal input using complex replacement and the ERF.

18.2 Introduction

The constant coefficient system $\mathbf{x}' = A\mathbf{x}$ has a nice conceptual solution in terms of the matrix exponential e^{At}. This matrix exponential is a square matrix whose derivative follows the usual rule for exponentials:
$$\frac{de^{At}}{dt} = Ae^{At}.$$
So, as can be checked directly, the system $\mathbf{x}' = A\mathbf{x}$ has solution $\mathbf{x}(t) = e^{At}\mathbf{c}$, where $\mathbf{c}$ is a constant vector.

We'll use the diagonalization $A = S\Lambda S^{-1}$ to define the matrix exponential e^{At}. We will then use it to give another way of presenting the solutions to $\mathbf{x}' = A\mathbf{x}$.

After that, we will turn our attention to inhomogeneous linear systems of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t). \tag{I}$$

As usual, $\mathbf{x}$ is a column vector of (unknown) functions, A is a square constant matrix and the input $\mathbf{F}(t)$ is a column vector. As you might expect, when $\mathbf{F}(t)$ is exponential or sinusoidal we will have an exponential or sinusoidal resposnse formula. Unlike for ordinary differential equations, these formulas are not worth memorizing. It will turn out to be easier to rederive them as needed.

18.3   Matrix Exponential

In 18.03 we use the exponential function all the time. Its main property is that it helps us solve differential equations.

Example 18.1. Solve $x' = ax$

Solution: $x(t) = x(0) \, e^{at}$.

We are going to define the matrix exponential. There are several ways to do this. Since this is a differential equations class, let's define it as the solution to a DE. Then we will see various ways to compute and use it.

Definition. For any square matrix A, the matrix exponential e^{At} is the matrix of functions that satisfies the initial value problem

$$\frac{dB(t)}{dt} = A \cdot B(t), \qquad B(0) = I.$$

Note. We could also have defined e^{At} using the Taylor series for e^x

$$e^{At} = I + tA + \frac{t^2 A^2}{2} + \frac{t^3 A^3}{3!} + \dots$$

Either definition gives the same answer.

We can now list several properties of the matrix exponential.
1. The initial value problem $\mathbf{x}' = A\mathbf{x}$ with initial value $\mathbf{x}(0) = \mathbf{b}$ has solution $e^{At}\mathbf{b}$.

2. If $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ then $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$.

3. If $A = S\Lambda S^{-1}$ is the diagonalization of A then

$$e^{At} = Se^{\Lambda t}S^{-1}$$

4. $e^{A(s+t)} = e^{As}e^{At}$.

5. Definition. e^{At} is called a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$

Warning: Because matrix multiplication does not commute, it is **not generally true** that $e^A e^B$ is the same as e^{A+B}. They are the same only in special cases.

Proofs. Here are proofs of these facts.

1. We need to verify that $\mathbf{x}(t) = e^{At}\mathbf{b}$ satisfies the IVP. This follows directly from our definition of matrix exponential:

$$\mathbf{x}'(t) = \frac{de^{At}\mathbf{b}}{dt} = A\mathbf{e^{At}b} = A\mathbf{x}(t).$$

2. $\dfrac{d}{dt}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \Lambda e^{\Lambda t}.$ QED

3. We need to show that $\dfrac{d}{dt}Se^{\Lambda t}S^{-1} = ASe^{\Lambda t}S^{-1}$. We do this by computing both sides and seeing that they are equal:

Since S is constant, the left-hand side of this equation is:

$$\frac{d}{dt}Se^{\Lambda t}S^{-1} = S\frac{de^{\Lambda t}}{dt}S^{-1} = S\Lambda e^{\Lambda t}S^{-1}.$$

Replacing A by its diagonalization, the right hand side of the equation is:

$$ASe^{\Lambda t}S^{-1} = S\Lambda S^{-1}Se^{\Lambda t}S^{-1} = S\Lambda\, e^{\Lambda t}S^{-1}.$$

The two sides are the same.   QED

4. This follows from the diagonalized form. To make the calculation explicit, we show it for the 2×2 case with eigenvalues λ_1, λ_2.

$$e^{As}e^{At} = Se^{\Lambda s}S^{-1}Se^{\Lambda t}S^{-1} = Se^{\Lambda s}e^{\Lambda t}S^{-1} = S\begin{bmatrix} e^{\lambda_1 s} & 0 \\ 0 & e^{\lambda_2 s}\end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{bmatrix}S^{-1}$$

$$= S\begin{bmatrix} e^{\lambda_1 (s+t)} & 0 \\ 0 & e^{\lambda_2(s+t)}\end{bmatrix}S^{-1} = Se^{\Lambda(s+t)}S^{-1} = e^{A(s+t)}. \quad \blacksquare$$

Example 18.2. Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Solution: We know the answer is $\mathbf{x} = e^{At}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

We also know A has eigenvalues 7, 1 and corresponding eigenvectors $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We can rewrite $\mathbf{x}(t) = e^{At}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ as

$$\mathbf{x}(t) = Se^{\Lambda t}\, S^{-1}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{7t} & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad (*)$$

$$= \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{7t} & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 8/6 \\ -22//6 \end{bmatrix} = \frac{8}{6}e^{7t}\begin{bmatrix} 5 \\ 1 \end{bmatrix} - \frac{22}{6}e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

As a general rule, the line marked with the $(*)$ is a fine answer to this question.

18.4   Exponential response formula (ERF)

Exponential response formula. For a constant matrix A and a constant vector $\mathbf{k}$ the DE

$$\mathbf{x}' = A\mathbf{x} + e^{at}\mathbf{k}$$

has a particular solution:

$$\mathbf{x_p}(t) = e^{at}(aI - A)^{-1}\mathbf{k}$$

This formula is valid as long as $aI - A$ is invertible, i.e., as long as a is not an eigenvalue of A.

Proof. Not surprisingly, we discover this formula by the method of optimism. We try a solution of the form $\mathbf{x_p}(t) = e^{at}\mathbf{v}$, where $\mathbf{v}$ is a constant vector.

Plug the guess into the DE and solve for **v**:

$$\mathbf{x_p}' = ae^{at}\mathbf{v} = e^{at}A\mathbf{v} + e^{at}\mathbf{k} \implies (aI - A)\mathbf{v} = \mathbf{k} \implies \mathbf{v} = (aI - A)^{-1}\mathbf{k}.$$

Thus we have found a particular solution $\mathbf{x_p}(t) = e^{at}\mathbf{v} = e^{at}(aI - A)^{-1}\mathbf{k}$.   QED

Example 18.3. Find the general solution to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix}$.

Solution: For ease of notation we rewrite the equation as $\mathbf{x}' = A\mathbf{x} + e^{2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.   The exponential response formula gives us a particular solution

$$\mathbf{x_p}(t) = e^{2t}(2I - A)^{-1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = e^{2t}\begin{bmatrix} -4 & -5 \\ -1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = -\frac{e^{2t}}{5}\begin{bmatrix} 0 & 5 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = -\frac{1}{5}e^{2t}\begin{bmatrix} 15 \\ -11 \end{bmatrix}$$

We know from previous topics that the general homogeneous equation is

$$\mathbf{x_h}(t) = c_1 e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

By superposition the general solution to the system is $\mathbf{x}(t) = \mathbf{x_p}(t) + \mathbf{x_h}(t)$.

Example 18.4. Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^{2t} \\ 5e^{3t} \end{bmatrix}$.

Solution: Write the input as $e^{2t}\begin{bmatrix} 3 \\ 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Now you can find a particular solution to the equation for a each input term and then use superposition.

There are more examples in the next section.

18.5   Exponential response formula examples

Example 18.5. Find the general solution to $\begin{aligned} x' &= 3x - y + e^{2t} \\ y' &= 4x - y - e^{2t} \end{aligned}$

Solution: In matrix form the equation is $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}\mathbf{x} + e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The exponential response formula tells us a particular solution is

$$\mathbf{x_p}(t) = e^{2t}(2I - A)^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{2t}\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{2t}\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{2t}\begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

We'll let you verify the calculation of the inverse. Likewise we'll let you find the homogeneous solution needed for the general solution.

Example 18.6. Find a particular solution to $\begin{aligned} x' &= 3x - y + 3 \\ y' &= 4x - y + 2 \end{aligned}$

Solution: Note that we could get our solution using the exponential response formula, where the exponent is $a = 0$. Instead, we'll just say that we're guessing a constant solution and solve for its exact value.

Try $\mathbf{x} = \mathbf{v}$. Substitution into the DE gives $\mathbf{x}' = 0 = A\mathbf{v} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

So, $\mathbf{v} = -A^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = -\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. That is $\mathbf{x_p}(t) = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

Again, we'll let you verify the calculation of the inverse.

Example 18.7. Find a particular solution to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix}$.

Solution: To use the exponential response formula, we first need to use complex replacement. The complexified equation is

$$\mathbf{z}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{z} + e^{it} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{where } \mathbf{x} = \text{Re}(\mathbf{z}).$$

Now we compute the inverse to prepare for the exponential response formula:

$$(iI - A)^{-1} = \begin{bmatrix} -1+i & -2 \\ -2 & -1+i \end{bmatrix}^{-1} = \frac{1}{-2i-4} \begin{bmatrix} -1+i & 2 \\ 2 & -1+i \end{bmatrix}$$

So, $\mathbf{z_p}(t) = e^{it}(iI - A)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{-2i-4} e^{it} \begin{bmatrix} -1+i & 2 \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2i+4} e^{it} \begin{bmatrix} 1-i \\ -2 \end{bmatrix}$.

To find the real part of $\mathbf{z_p}$, we work in polar coordinates. First we write the various complex numbers in polar form:

$2i + 4 = 2\sqrt{5} e^{i\phi_1}$, where $\boxed{\phi_1 = \text{Arg}(2i+4) = \tan^{-1}(1/2)}$ in the first quadrant.

Likewise $1 - i = \sqrt{2} e^{i\phi_2}$, where $\boxed{\phi_2 = -\pi/4}$.

So, $\mathbf{z_p}(t) = \frac{-e^{it}}{2\sqrt{5} e^{i\phi_1}} \begin{bmatrix} \sqrt{2} e^{i\phi_2} \\ -2 \end{bmatrix} = -\frac{1}{2\sqrt{5}} \begin{bmatrix} \sqrt{2} e^{i(t+\phi_2-\phi_1)} \\ -2 e^{i(t-\phi_1)} \end{bmatrix}$.

Taking the real part:

$$\boxed{\mathbf{x_p}(t) = \text{Re}(\mathbf{z_p}) = \begin{bmatrix} \sqrt{2}\cos(t + \phi_2 - \phi_1) \\ -2\cos(t - \phi_1) \end{bmatrix}}$$

Here is the same calculation in rectangular coordinates. I think the arithmetic is more error prone and the answer is harder to interpret.

$$\frac{1}{2i+4} \begin{bmatrix} 1-i \\ -2 \end{bmatrix} = \frac{4-2i}{20} \begin{bmatrix} 1-i \\ -2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1-3i \\ -4+2i \end{bmatrix}.$$

So, $\mathbf{z_p}(t) = \frac{1}{10}(\cos(t)+i\sin(t)) \begin{bmatrix} 1-3i \\ -4+2i \end{bmatrix} = \frac{1}{10} \begin{bmatrix} \cos(t) + 3\sin(t) + i(\sin(t) - 3\cos(t)) \\ -4\cos(t) - 2\sin(t) + i(-4\sin(t) + 2\cos(t)) \end{bmatrix}$.

Thus, $\mathbf{x_p}(t) = \text{Re}(\mathbf{z_p}(t)) = \frac{1}{10} \begin{bmatrix} \cos(t) + 3\sin(t) \\ -4\cos(t) - 2\sin(t) \end{bmatrix}$.

MIT OpenCourseWare

https://ocw.mit.edu

ES.1803 Differential Equations

Spring 2024