ES.1803 Topic 19 Notes

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19 Fundamental matrix, variation of parameters

This topic is no longer on the syllabus. We post these notes for anyone who is interested.

19.1 Goals

1. Be able to recognize a linear non-constant coefficient system of differential equations.
2. Know the definition and basic properties of a fundamental matrix for such a system.
3. Be able to use the matrix exponential as a fundamental matrix for a constant coefficient linear system.
4. Be able to use the variation of paramters formula to solve a (nonconstant) coefficient linear inhomogeneous system.
5. Be able to use Euler's method to approximate the solution to a system of first-order equations.

19.2 Introduction

So far we have focused on homogeneous, constant coefficient linear systems. We now want to think about systems with input or with non-constant coefficients. So in this topic we will consider general linear systems of differential equations. That is, equations of the following form.

$$\mathbf{x}' = A(t)\mathbf{x} \qquad \text{(homogeneous)} \qquad \text{(H)}$$

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t) \qquad \text{(inhomogeneous)} \qquad \text{(I)}$$

Here $\mathbf{x}(t)$ is a vector valued function, e.g., $(x(t), y(t), z(t))^T$, $A(t)$ is an $n \times n$ matrix called the coefficient matrix and $\mathbf{F}(t)$ is called the (mathematical) input to the system.

As usual, solving the system means finding the unknown vector valued function $\mathbf{x}(t)$.

A main point in this topic is to introduce the fundamental matrix, $\Phi(t)$, for a linear system of DEs. This will allow us to state the essential properties of these systems in a concise and elegant way. The fundamental matrix is available for any linear system. We will see that the matrix exponential e^{At}, introduced in a previous topic, is a fundamental matrix for the constant coefficient system $\mathbf{x}' = A\mathbf{x}$.

Next, we will look at linear equations with arbitrary input. This will lead to the variation of parameters formula for the solution. This is a beautiful formula, which uses the fundamental matrix. Since it involves integrals and can be painful or difficult to apply, we will use it as a last resort to find solutions to equations with nonconstant coefficients or unusual input.

We will conclude with a small section showing that Euler's method works for systems of first-order equations in exactly the same way as for ordinary first-order differential equations.

We start by going over the familiar ideas of linearity and existence and uniqueness.

19.3 Linearity/Superposition

As always, linear systems satisfy superposition principles. We restate them in the forms we like to use.

1. If $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to Equation (H), then so is $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$

Proof. $\mathbf{x}' = c_1\mathbf{x_1}' + c_2\mathbf{x_2}' = c_1 A\mathbf{x_1} + c_2 A\mathbf{x_2} = A(c_1\mathbf{x_1} + c_2\mathbf{x_2}) = A\mathbf{x}$.

2. If $\mathbf{x_h}$ is a solution to Equation (H) and $\mathbf{x_p}$ is a solution to Equation (I) then $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$ is also a solution to Equation (I).

Proof. $\mathbf{x}' = \mathbf{x_p}' + \mathbf{x_h}' = A\mathbf{x_p} + \mathbf{F} + A\mathbf{x_h} = A(\mathbf{x_p} + \mathbf{x_h}) + \mathbf{F} = A\mathbf{x} + \mathbf{F}$.

3. If $\mathbf{x_1}' = A\mathbf{x_1} + \mathbf{F_1}$ and $\mathbf{x_2}' = A\mathbf{x_2} + \mathbf{F_2}$ then $\mathbf{x_1} + \mathbf{x_2}$ satisfies $\mathbf{x}' = A\mathbf{x} + \mathbf{F_1} + \mathbf{F_2}$

That is, superposition of inputs leads to superposition of outputs.

Proof. Just the same.

19.4 Existence and uniqueness theorem

As we've done for other types of equations, we state an existence and uniqueness theorem so that we can be sure that we have found all the solutions when we use the $x(t) = x_p(t) + x_h(t)$ paradigm.

Consider the initial value problem:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t), \quad \mathbf{x}(t_0) = \mathbf{x_0} \tag{IVP}$$

The existence and uniqueness theorem says that there is exactly one solution to this equation.

Theorem. (existence and uniqueness) If $A(t)$ and $\mathbf{F}(t)$ are continuous then there exists a unique solution to the equation (IVP).

The next example illustrates that this new version of the existence and uniqueness theorem agrees with our old version for second-order linear equations.

Example 19.1. Consider the IVP $x'' + tx' + t^2x = t^3$; $x(0) = 1$, $x'(0) = 3$.

Converting this DE to a system using $y = x'$, we get:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -t^2 & -t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ t^3 \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

More abstractly we can write this as: $\mathbf{x}' = A\mathbf{x} + \mathbf{F}$; $\mathbf{x}(0) = \begin{bmatrix} 1 & 3 \end{bmatrix}^{\mathrm{T}}$

Since $A(t)$ and $\mathbf{F}(t)$ are continuous the existence and uniqueness for systems says there is a unique solution to the system. Now, $x(t)$ is the first entry in this solution, so there is also a unique solution to the original IVP.

Note. Previously, we had an existence and uniqueness theorem for ordinary differential equations which said exactly the same thing.

19.5 Fundamental matrix

This is an elegant bookkeeping technique which will make calculations and theorem statements much nicer. Consider the linear homogeneous system

$$\mathbf{x}' = A(t)\mathbf{x} \tag{H}$$

Suppose it is an $n \times n$ system and that we have n independent solutions $\mathbf{x_1}, \ldots \mathbf{x_n}$. We define the fundamental matrix as the matrix with columns $\mathbf{x_1}, \ldots, \mathbf{x_n}$, i.e.

$$\Phi(t) = \begin{bmatrix} \mathbf{x_1}(t) \ \mathbf{x_2}(t) \ \ldots \ \mathbf{x_n}(t) \end{bmatrix}.$$

Example 19.2. Consider the system $\mathbf{x}' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \mathbf{x}$.

(a) Find a fundamental matrix for this system.

(b) Use the fundamental matrix to give the general solution to this system.

(c) Find the solution with initial conditions $\mathbf{x}(t_o) = \mathbf{b}$.

Solution: (a) We've used this coefficient matrix many times. We know two independent solutions to the system are

$$\mathbf{x_1} = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{x_2} = e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

So a fundamental matrix is $\Phi(t) = \begin{bmatrix} e^t & 5e^{7t} \\ -e^t & e^{7t} \end{bmatrix}$.

(b) The general solution is

$$\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1 \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + c_2 \begin{bmatrix} 5e^{7t} \\ e^{7t} \end{bmatrix} = \Phi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

(The last expression follows because matrix multiplication is a linear combination of the columns of Φ.)

(c) Now, we can use this to find the solution to the IVP with initial conditions $\mathbf{x}(t_0) = \mathbf{b}$.

$$\mathbf{x}(t) = \Phi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Rightarrow \quad \Phi(t_0) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{b} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Phi^{-1}(t_0)\mathbf{b}.$$

This is valid provided $\Phi^{-1}(t_0)$ exists. We will show this below.

19.5.1 Properties of the fundamental matrix

We have the following important properties of the fundamental matrix Φ.

1. $\Phi'(t) = A(t)\Phi(t)$ i.e., Φ satisfies Equation (H).
2. If $\mathbf{c}$ is a column vector, then $\Phi(t) \cdot \mathbf{c} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + ... + c_n\mathbf{x_n}$.
3. If $A(t)$ is continuous, then $W(t) = |\Phi(t)| \neq 0$ equivalently $\Phi^{-1}(t)$ exists. (We call $W(t)$ the Wronskian of $\mathbf{x_1}, \ldots, \mathbf{x_n}$.)

Proof. (1) Before proving this, we note the following property of matrix multiplication: if B has columns $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}$ then

$$AB = \begin{bmatrix} A\mathbf{b_1} & A\mathbf{b_2} & \ldots & A\mathbf{b_n} \end{bmatrix}.$$

You should make sure you understand this. (If it is confusing, work out a simple numerical example with an eye to understanding this property.)

Now (1) follows easily from this property:

$$\Phi'(t) = \begin{bmatrix} \mathbf{x}'_1 & \mathbf{x}'_2 & \ldots & \mathbf{x}'_n \end{bmatrix} = \begin{bmatrix} A\mathbf{x_1} & A\mathbf{x_2} & \ldots & A\mathbf{x_n} \end{bmatrix} = A \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \ldots & \mathbf{x_n} \end{bmatrix} = A(t)\Phi(t).$$

The second equality above follows because the $\mathbf{x_j}$ are solutions to Equation (H). The third equality is the property of matrix multiplication discussed just above.

(2) This is just a property of matrix multiplication.

(3) We will prove this by contradiction, i.e., we'll assume that for some t_0, $W(t_0) = 0$ and show that this contradicts the existence and uniqueness theorem. So suppose that $W(t_0) = 0$. This implies that $\Phi(t_0)$ has a nontrivial null space. Let $\mathbf{c} \neq 0$ be a nontrivial null vector. The contradiction is that now there are two solutions with $\mathbf{x}(t_0) = \mathbf{0}$. That is, both

$$\mathbf{x_1}(t) \equiv 0 \quad \text{and} \quad \mathbf{x_2}(t) = \Phi(t)\mathbf{c}$$

are 0 at $t = t_0$. This contradiction means that our assumption that $W(t_0) = 0$ must be false. QED

Example 19.3. Consider the system $\mathbf{x}' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \mathbf{x}$ from Example 19.2. Show that its Wronskian is never 0.

Solution: In example 19.2 we found the fundamental matrix $\Phi(t) = \begin{bmatrix} e^t & 5e^{7t} \\ -e^t & e^{7t} \end{bmatrix}$

So the Wronskian is $W(t) = |\Phi(t)| = e^{8t} + 5e^{8t} = 6e^{8t}$, which is never 0.

Example 19.4. Again, consider the system $\mathbf{x}' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \mathbf{x}$. Let A be the coefficient matrix. Show that the matrix exponential e^{At} is a fundamental matrix and compute its Wronskian.

Solution: To show e^{At} is a fundamental matrix, we need to show that every solution can be written as $e^{At}\mathbf{c}$ for some constant vector $\mathbf{c}$. This was shown in the Topic 18 notes.

To compute the Wronskian we use the diagonalized form of A:

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}.$$

So,

$$W(t) = \det(e^{At}) = \det(Se^{\Lambda t}S^{-1}) = \det(e^{\Lambda t}) = \det\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{7t} \end{bmatrix}\right) = e^{8t} \neq 0.$$

19.5.2 The Wronskian of n solutions

In the above we assumed that the solutions were independent. Even if they are not, we can still define the Wronskian: Suppose $\mathbf{x_1}$, ... $\mathbf{x_n}$ are solutions to Equation (H). We call the determinant $W(t) = \det \begin{bmatrix} \mathbf{x_1} & \dots & \mathbf{x_n} \end{bmatrix}$ the Wronskian of these solutions. If $A(t)$ is continuous then the existence and uniquenss theorem implies:

(i) $W(t)$ is either always 0 or never 0.

(ii) $W(t) \neq 0 \Leftrightarrow \mathbf{x_1}, \dots, \mathbf{x_n}$ are independent.

(iii) $W(t) \neq 0 \Leftrightarrow \Phi = \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \dots & \mathbf{x_n} \end{bmatrix}$ is a fundamental matrix.

Conclusion: we can use the Wronskian to test for independence.

Example 19.5. Consider $x'' + p(t)x' + q(t)x = 0$, with solutions x_1, x_2. Convert this to a first-order system. Then give two solutions to the system and compute their Wronskian.

Solution: The companion system is found by setting $y = x'$. Thus the solutions x_1 and x_2 of the ordinary differential equation become the solutions $\mathbf{x_1} = \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} x_2 \\ x_2' \end{bmatrix}$ of the companion system. Using the definition of the Wronskian we have

$$W(t) = \det \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} = x_1 x_2' - x_1' x_2.$$

19.6 Variation of parameters formula

For the general, not necessarily constant coefficient, linear inhomogeneous system (I) we cannot use constant coefficient techniques like the ERF. For those cases where we have no other technique, we can try to use the variation of parameters formula. Since it involves integration, matrix inverses and matrix multiplication, it is our last choice when trying to solve an equation. Nonetheless, sometimes it's the only method available. In addition, the derivation of the formula is really very pretty.

Suppose we have a fundamental matrix $\Phi(t)$ for the *homogeneous* linear equation

$$\mathbf{x}' = A(t)\mathbf{x} \tag{H}$$

Remember this means that Φ has columns which are independent solutions to (H).

Now suppose we want to solve

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t). \tag{I}$$

Theorem. The general solution to equation (I) is given by the variation of parameters formula

$$\mathbf{x}(t) = \Phi(t) \cdot \left(\int \Phi(t)^{-1} \cdot \mathbf{F}(t)\, dt + \mathbf{C} \right).$$

Proof. We will use a form of the method of optimism to derive this formula.

We know the general homogeneous solution is $\mathbf{x}(t) = \Phi(t) \cdot \mathbf{c}$ for a constant vector $\mathbf{c}$. The vector $\mathbf{c}$ is called a parameter. Variation of parameters is an old-fashioned way of saying let's optimistically make it a (dependent) variable $\mathbf{u}(t)$. So we try a solution of the form

$\mathbf{x}(t) = \Phi(t) \cdot \mathbf{u}(t)$. The function $\mathbf{u}(t)$ is unknown. To find it, we substitute our guess into (I) and see where the algebra leads us:

$$\Phi' \cdot \mathbf{u} + \Phi \cdot \mathbf{u}' = A\Phi \cdot \mathbf{u} + \mathbf{F}$$

So, (don't forget $\Phi' = A\Phi$.)

$$A\Phi \cdot \mathbf{u} + \Phi \cdot \mathbf{u}' = A\Phi \cdot \mathbf{u} + \mathbf{F} \quad \Rightarrow \quad \Phi \cdot \mathbf{u}' = \mathbf{F}.$$

This last equation is easy to solve:

$$\mathbf{u}' = \Phi^{-1} \cdot \mathbf{F} \quad \Rightarrow \quad \mathbf{u}(t) = \int \Phi^{-1}(t) \cdot \mathbf{F}(t)\, dt + \mathbf{C}.$$

Finally, we take this formula for $\mathbf{u}(t)$ and use it in our trial solution:

$$\mathbf{x}(t) = \Phi(t) \cdot \mathbf{u}(t) = \Phi(t) \cdot \left(\int \Phi(t)^{-1} \cdot \mathbf{F}(t)\, dt + \mathbf{C} \right). \quad \blacksquare$$

Remark. Note that the variation of parameters formula assumes you know the general homogeneous solution. It gives no help in finding this solution.

Example 19.6. Use the variation of parameters formula to solve

$$\mathbf{x}' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ e^{5t} \end{bmatrix}.$$

Note. We retiterate that using the ERF is the preferred method of solving this equation. We use the variation of parameters formula here for practice.

Solution: Let's introduce some notation to save typing: $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$, $\mathbf{F} = \begin{bmatrix} 1 \\ t \end{bmatrix}$.

We know a fundamental matrix from an earlier example: $\Phi(t) = \begin{bmatrix} e^t & 5e^{7t} \\ -e^t & e^{7t} \end{bmatrix}$. So, $\Phi^{-1}(t) = \frac{e^{-8t}}{6} \begin{bmatrix} e^{7t} & -5e^{7t} \\ e^t & e^t \end{bmatrix}$. Calculating with the variation of parameters we get

$$\begin{aligned}
\mathbf{x} &= \Phi(t) \int \Phi^{-1}(t) \cdot \mathbf{F}(t)\, dt \\
&= \Phi(t) \int \frac{e^{-8t}}{6} \begin{bmatrix} e^{7t} & -5e^{7t} \\ e^t & e^t \end{bmatrix} \cdot \begin{bmatrix} e^t \\ e^{5t} \end{bmatrix} dt \\
&= \Phi(t) \int \frac{1}{6} \begin{bmatrix} 1 - 5e^{4t} \\ e^{-6t} + e^{-2t} \end{bmatrix} dt \\
&= \frac{1}{6} \Phi(t) \begin{bmatrix} t - \frac{5}{4}e^{4t} + c_1 \\ -\frac{1}{6}e^{-6t} - \frac{1}{2}e^{-2t} + c_2 \end{bmatrix} \\
&= \frac{1}{6} \begin{bmatrix} te^t - \frac{5}{4}e^{5t} - \frac{5}{6}e^t - \frac{5}{2}e^{5t} + c_1e^t + 5c_2e^{7t} \\ -te^t + \frac{5}{4}e^{5t} - \frac{1}{6}e^t - \frac{1}{2}e^{5t} - c_1e^t + c_2e^{7t} \end{bmatrix} \\
&= \frac{1}{6} \left(te^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{5t} \begin{bmatrix} -15/4 \\ 3/4 \end{bmatrix} + e^t \begin{bmatrix} -5/6 \\ -1/6 \end{bmatrix} + c_1e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right).
\end{aligned}$$

Notice the homogeneous solution appearing with the constants of integration.

19.6.1 Definite integral version of variation of parameters

The equation (I) with initial condition $\mathbf{x}(t_0) = \mathbf{b}$ has definite integral solution

$$\mathbf{x}(t) = \Phi(t)\left(\int_{t_0}^{t} \Phi^{-1}(u)\cdot\mathbf{F}(u)\,du + \mathbf{C}\right) \text{ where } \mathbf{C} = \Phi^{-1}(t_0)\cdot\mathbf{b}.$$

19.7 Euler's method

Consider a first-order system with initial conditions:

$$\mathbf{x} = \mathbf{F}(\mathbf{x}, t), \qquad \mathbf{x}(t_0) = \mathbf{x_0}.$$

Euler's method for ordinary first-order DEs works without any change for this first-order systems. That is, fix a stepsize h. Then, the step from $(\mathbf{x_n}, t_n)$ to $(\mathbf{x_{n+1}}, t_{n+1})$ is given by

$$\mathbf{m} = \mathbf{F}(\mathbf{x_n}, t_n) \quad \Rightarrow \quad \mathbf{x_{n+1}} = \mathbf{x_n} + h\mathbf{m}, \quad t_{n+1} = t_n + h.$$

Just as for ordinary DEs, there are other, better, algorithms for choosing $\mathbf{m}$ or varying h.

Example 19.7. Consider $\begin{bmatrix} x' \\ y' \end{bmatrix} = t\begin{bmatrix} y \\ x \end{bmatrix}$, $\mathbf{x}(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and use $h = 0.5$ to estimate $\mathbf{x}(2)$.

Solution:

n	t_n	$\mathbf{x_n}$	$\mathbf{m} = \mathbf{F}(\mathbf{x_n}, t_n)$
0	1.0	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
1	1.5	$\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$	$\begin{bmatrix} 0.75 \\ 1.5 \end{bmatrix}$
2	2.0	$\begin{bmatrix} 1.375 \\ 1.25 \end{bmatrix}$	

So, $\mathbf{x}(2) \approx \begin{bmatrix} 1.375 \\ 11.25 \end{bmatrix}$.