

ES.1803 Topic 2 Notes

Jeremy Orloff

2 Linear systems: input-response models

2.1 Goals

1. Be able to classify a first-order differential equation as linear or nonlinear.
2. Be able to put a first-order linear DE into standard form.
3. Be able to use the variation of parameters formula to solve a first-order linear DE.
4. Be able to explain why the superposition principle holds for first-order linear DEs
5. Be able to use the superposition principle to solve a first-order linear DE by breaking the input into pieces.

2.2 Linear first-order differential equations

To start with we will define linear first-order equations by their form. Soon we will understand them by their properties. In particular, you should be on the lookout for the statement of the [superposition principle](#) and in later topics for the conceptual definition of [linearity](#).

2.2.1 General and standard form of first-order linear differential equations

Definition. The [general first-order linear](#) differential equation has the form

$$A(t) \frac{dy}{dt} + B(t)y(t) = C(t). \quad (1)$$

As long as $A(t) \neq 0$ we can simplify the equation by dividing by $A(t)$. This gives the [standard form](#) of a first-order linear differential equation.

$$\frac{dy}{dt} + p(t)y(t) = q(t). \quad (2)$$

Most often when working with linear DEs we will need to put it in the standard form in Equation 2.

2.2.2 Terminology and notation

The functions $A(t)$, $B(t)$ in Equation 1 and $p(t)$ in Equation 2 are called the [coefficients](#) of the differential equation. If A and B (or p) are constants, i.e., do not depend on the variable t , then we say the equation is a [constant coefficient differential equation](#).

Notice that the functions $C(t)$ or $q(t)$ on the right-hand side of the equations are not called coefficients and do not have to be constant, even in a constant coefficient DE.

2.2.3 Homogeneous/inhomogeneous

If $C(t) = 0$ in Equation 1 then the resulting equation:

$$A(t)y' + B(t)y = 0$$

is called **homogeneous**. Otherwise the equation is called **inhomogeneous**.

Note: Homogeneous is not the same word as homogenous (or homogenized). In homogeneous the syllable 'ge' is pronounced with a long e and is stressed, while the syllable 'mo' is stressed in homogenous.

2.2.4 Identifying first-order linear equations

The Equations 1 and 2 have the form that y' and y occur separately and only as first powers.

Example 2.1. The following differential equations are all linear:

Linear: $y' = ky$; $y' + e^{\sin(t)}y = t^2$; $y' + t^2y = t^3$.

And the following are all non-linear:

Non-linear: $y' + y^2 = t$; $(y')^2 + y = t$; $y'y = t$.

Notice that the coefficient functions in a linear DE are not restricted in any way, but that y and y' never occur in the same term and only have first powers.

Example 2.2. Modeling a population of oryx. A population of oryx has a natural growth rate k in units of 1/year and they are harvested at a constant rate of h oryxes/year. Construct a first-order differential equation modeling the population over time.



An Oryx *gazella*, also known as a Gemsbok
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Solution: Let $y(t)$ be the oryx population. By *natural growth rate* we mean that without any outside influences population grows at a rate proportional to itself, i.e., $y' = ky$. The harvesting changes the growth rate by removing oryx at the rate h . Combining the two rates we have

$$y' = ky - h.$$

This is a first-order linear DE. In standard form it reads

$$y' - ky = -h.$$

2.3 Solving first-order linear equations

2.3.1 The variation of parameters formula

We start by giving a formula for the solution to a first-order linear DE in standard form. The differential equation $y' + p(t)y = q(t)$ has solution

$$y(t) = y_h(t) \int \frac{q(t)}{y_h(t)} dt + Cy_h(t), \text{ where } y_h(t) = e^{-\int p(t) dt}. \quad (3)$$

The function $y_h(t)$ is the solution to the [associated homogeneous equation](#):

$$y'_h + p(t)y_h = 0.$$

Notes: **1.** As usual for a first-order DE, the solution is a one parameter family of functions.

2. The formula in Equation 3 is called the [variation of parameters formula](#). The reason for the name comes from the method of deriving it that we give in the last section of this topic's notes.

Warning: The variation of parameters formula is quite beautiful, but don't be seduced into using it in every situation. Because it involves integration it is, generally speaking, our method of **last resort**. When we focus on constant coefficient equations we will learn easier and more informative techniques.

2.3.2 Examples

Example 2.3. Solve $y' + ky = k$, where k is a constant.

Solution: In this case $p(t) = k$ is a constant. The homogeneous solution is

$$y_h(t) = e^{-\int k dt} = e^{-kt}.$$

Therefore, the general solution to the DE is

$$\begin{aligned} y(t) &= y_h(t) \int q(t)/y_h(t) dt + Cy_h(t) = e^{-kt} \int \frac{k}{e^{-kt}} dt + Ce^{-kt} \\ &= e^{-kt} \int ke^{kt} dt + Ce^{-kt} = e^{-kt} \cdot e^{kt} + Ce^{-kt} = \boxed{1 + Ce^{-kt}} \end{aligned}$$

(Again: don't get too attached to this technique, later we will learn better techniques for solving constant coefficient equations.)

Example 2.4. Solve $y' + ky = kt$, where k is a constant.

Solution: y_h is the same as in the previous example. Therefore,

$$y(t) = e^{-kt} \int kte^{kt} dt + Ce^{-kt} = \left(t - \frac{1}{k}\right) + Ce^{-kt}.$$

(We computed this integral using integration by parts.)

2.4 Superposition principle

We start by defining the terms superposition, input and output.

Superposition is a fancy way of describing adding together multiples of two functions.

Examples: 1. The function $q(t) = 3t + 4t^2$ is a superposition of t and t^2 .

2. If $q_1(t)$ and $q_2(t)$ are functions then $q(t) = 3q_1 + 4q_2$ is a superposition of q_1 and q_2 .

3. If $q_1(t)$ and $q_2(t)$ are functions and c_1 and c_2 are constants then $q(t) = c_1q_1 + c_2q_2$ is a superposition of q_1 and q_2 .

We will also say that $q = c_1q_1 + c_2q_2$ is a **linear combination** of q_1 and q_2 .

Suppose we have the first-order linear differential equation

$$y' + p(t)y = q(t). \quad (4)$$

We will often call the $q(t)$ the **input**. We will then call $y(t)$ the **output** of the system to the input q . Of course, $y(t)$ is nothing more than the solution to the DE. In Topic 3 we will expand on the notions of input and output.

The superposition principle is easy but extremely important! It concerns the linear DE in Equation 4 with different inputs $q = q_1$, $q = q_2$ and $q = c_1q_1 + c_2q_2$.

Superposition principle. If

$$y_1 \text{ is a solution of the DE } y' + p(t)y = q_1(t)$$

and

$$y_2 \text{ is a solution of the DE } y' + p(t)y = q_2(t)$$

then for any constants c_1, c_2 we have

$$c_1y_1 + c_2y_2 \text{ is a solution of the DE } y' + p(t)y = c_1q_1(t) + c_2q_2(t).$$

Important note: Notice that the coefficient $p(t)$ is the same for all the DEs.

In words the superposition principle says: For first-order linear DEs

If the input q_1 has output y_1 and the input q_2 has output y_2 then the input $c_1q_1 + c_2q_2$ has output $c_1y_1 + c_2y_2$

An even simpler formulation is:

For linear DEs superposition of inputs gives superposition of outputs.

2.4.1 Proof of the superposition principle

First note that saying y_1 is a solution to $y' + py = q_1$ simply means $y_1' + py_1 = q_1$ and likewise for y_2 .

To prove the superposition principle we have to verify that $y = c_1y_1 + c_2y_2$ is indeed a solution to $y' + py = c_1q_1 + c_2q_2$. We do this by substitution:

$$\begin{aligned} y' + py &= (c_1y_1 + c_2y_2)' + p(c_1y_1 + c_2y_2) \\ &= c_1y_1' + c_2y_2' + c_1py_1 + c_2py_2 \\ &= c_1(y_1' + py_1) + c_2(y_2' + py_2) \\ &= c_1q_1 + c_2q_2. \end{aligned}$$

The last equality follows because of our assumption that $y_1' + py_1 = q_1$ and the similar assumption for y_2 . Now, looking at the first and last terms in this string of equalities we see that we have proved the superposition principle.

Example 2.5. Solve the linear DE $y' + 2y = 2 + 4t$.

Solution: You can easily check that

$$\begin{aligned} y' + 2y = 1 \text{ has solution } y_1 &= 1/2 + C_1 e^{-2t} \\ y' + 2y = t \text{ has solution } y_2 &= t/2 - \frac{1}{4} + C_2 e^{-2t} \end{aligned}$$

The input $2 + 4t$ is a linear combination of the inputs 1 and t , so by the superposition principle the solution to the DE is a linear combination of the outputs y_1, y_2

$$y = 2y_1 + 4y_2 = 1 + 2C_1 e^{-2t} + 2t - 1 + 4C_2 e^{-2t} = 2t + C e^{-2t}.$$

In the last equality we combined all of the coefficients of e^{-2t} into a single symbol C .

2.5 An extended example

Example 2.6. (*Heat diffusion.*) I put my root beer in a cooler, but after a while it still gets warm. Let's model its temperature using a differential equation.

Solution: First we need to name the function that measures the temperature:

$$\text{Let } x(t) = \text{root beer temperature at time } t.$$

The simplest model of this situation is Newton's law of cooling. It says that the rate the temperature of the root beer changes is proportional to the difference between the temperatures of the root beer and its environment. In symbols, let $E(t)$ be the temperature of the environment, then (using 'dot' notation)

$$\dot{x}(t) = -k(x(t) - E(t)),$$

where k is the constant of proportionality. Rearranging this equation it becomes

$$\dot{x} + kx = kE(t).$$

This is a first-order linear DE in standard form!

Example 2.7. Suppose the environment in the previous example is $E(t) = 60 + 6t$, where t is the time in hours from 10 AM. (So the temperature is rising linearly.) To be concrete, let's also assume $x(0) = 32^\circ\text{F}$ and $k = 1/3$. If I want to drink my root beer before it reaches 60°F how much time do I have?

Solution: Our strategy will be to first solve the initial value problem to find $x(t)$ and then use this to determine at what time $x(t)$ will be 60.

From the previous example we know that

$$\dot{x} + kx = kE \quad \text{so,} \quad \dot{x} + kx = 60k + 6kt.$$

We could apply the variation of parameters formula directly to this, but the superposition principle will do all the work for us. The input $60k + 6kt$ is a superposition of the inputs from

Examples 2.3 and 2.4. Therefore, the solution (output) is a superposition of the outputs from those examples. We know:

From Example 2.3: $\dot{x} + kx = k$ has solution $1 + Ce^{-kt}$.

From Example 2.4: $\dot{x} + kx = kt$ has solution $t - 1/k + Ce^{-kt}$.

Therefore, $\dot{x} + kx = 60k + 6kt$ has solution

$$x(t) = 60(1 + Ce^{-kt}) + 6(t - 1/k + Ce^{-kt}) = 60 + 6t - 6/k + \tilde{C}e^{-kt}.$$

Here we combined all the coefficients of e^{-kt} into one constant \tilde{C} . Now we set $k = 1/3$ to get

$$x(t) = 42 + 6t + \tilde{C}e^{-t/3}.$$

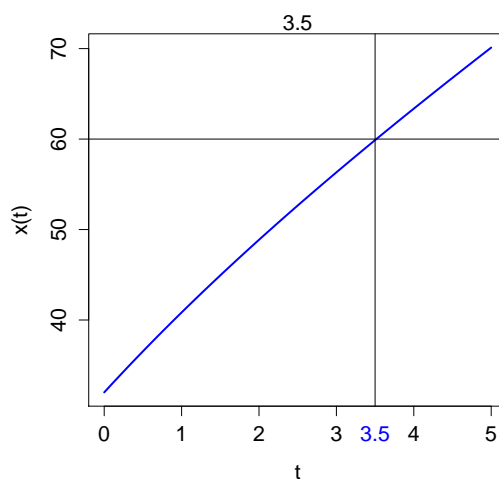
Finally, we use the initial condition to find \tilde{C} .

$$x(0) = 42 + \tilde{C} = 32, \text{ so } \tilde{C} = -10.$$

We've found the temperature of the root beer in my cooler is

$$x(t) = 42 + 6t - 10e^{-t/3}.$$

To answer the question we need to compute when $x(t) = 60$. Probably the easiest way to do this is to plot $x(t)$ and see where it crosses $x = 60$. We see this is at about $t = 3.5$. I have until about 1:30 pm to enjoy my drink.



Plot of $x(t)$: $x(t) = 60$ at approximately $t = 3.5$.

Remark redux: We hasten to point out once again that later we will learn faster and nicer techniques for solving equations like this. Techniques involving integration are generally last resorts, to be used when all else has failed.

2.6 Nonlinear equations don't satisfy the superposition principle

The superposition principle is the main reason we focus on linear differential equations. As we have seen in a few examples, it allows us to break the input of a linear equation

into pieces and construct the full solution out of the solutions to the pieces. In fact, the superposition principle only holds for linear DEs. We will illustrate this by showing it does not hold for a given nonlinear DE.

Example 2.8. Show that superposition does not hold for the nonlinear equation

$$y' + y^2 = q(t).$$

Solution: We can do this abstractly without actually solving the DE! Suppose $y_1' + y_1^2 = q_1$ and $y_2' + y_2^2 = q_2$. If superposition held for this equation then we would have

$$(y_1 + y_2)' + (y_1 + y_2)^2 = q_1 + q_2.$$

But it's easy to see this equation is not true:

$$\begin{aligned} (y_1 + y_2)' + (y_1 + y_2)^2 &= y_1' + y_2' + y_1^2 + 2y_1y_2 + y_2^2 \\ &= y_1' + y_1^2 + y_2' + y_2^2 + 2y_1y_2 \\ &= q_1 + q_2 + 2y_1y_2 \\ &\neq q_1 + q_2. \end{aligned}$$

What went wrong here? One way to say it is that superposition works for linear equations because the terms in the sum do not really interact. That is, in expressions like $(y_1 + y_2)' = y_1' + y_2'$ and $p(y_1 + y_2) = py_1 + py_2$ the effect on y_1 is exactly what it would be if y_2 was not there. On the other hand in the expression $(y_1 + y_2)^2 = y_1^2 + 2y_1y_2 + y_2^2$ the term $2y_1y_2$ represents an interaction between y_1 and y_2 . That is, the effect of squaring on y_1 is affected by the presence of y_2 .

2.7 Definite integral solutions to linear initial value problems

Consider the linear IVP

$$y' + p(t)y = q(t); \quad y(0) = y_0.$$

We can solve this equation by two methods.

Method 1: Use the variation of parameters formula in Equation 3 to find the general solution and then use the initial condition to solve for C .

Method 2: Use definite integrals in the variation of parameters formula to give the solution directly. We show how this is done: Take

$$y_h(t) = e^{-\int_{t_0}^t p(u) du}.$$

(This is chosen so that $y_h(0) = 1$.) Then

$$y(t) = y_h(t) \int_0^t \frac{q(u)}{y_h(u)} du + y_0 \cdot y_h(t).$$

Notes. 1. This form of the solution is well-suited for numerical computation.

2. We stated the problem with initial condition at $t = 0$, but we could have been more general and take $y(t_0) = y_0$.

Here is an example that illustrates both these points. Note that we don't compute the integral exactly, but we can still use the computer to compute approximate values of the solution.

Example 2.9. Solve the initial value problem $x^2y' + xy = \sin x$; $y(1) = y_0$.

Solution: First we need to convert the DE to standard form:

$$y' + \frac{1}{x}y = \frac{\sin x}{x^2}.$$

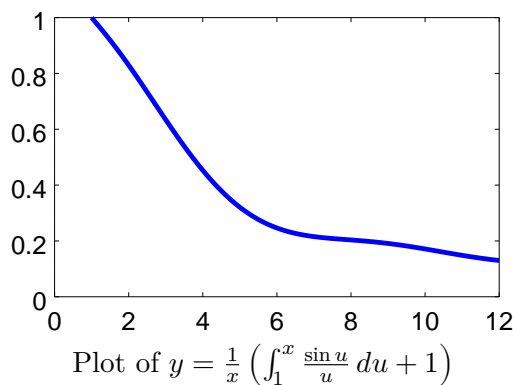
The homogeneous solution is

$$y_h = e^{-\int_1^x \frac{1}{u} du} = \frac{1}{x}.$$

So the variation of parameters formula gives

$$y = \frac{1}{x} \int_1^x \frac{\sin u}{u} du + \frac{y_0}{x}.$$

There is no closed form for the integral, but we can still use calculus to know a lot about this integral and to compute its value to any desired degree of accuracy. Here is a plot we made in Matlab (actually Octave) using its numerical integration function `quad()`. The initial value is $y_0 = 1$.



2.8 Proof of variation of parameters formula

(You are not responsible for knowing this yet. We will come back to it when we study systems of linear equations.)

One proof that the Equation 3 solves the DE in Equation 2 is by substitution. It's not difficult to plug the formula for $y(t)$ into the differential equation and check that it works. Of course, this is not a very satisfying proof because it fails to answer the question of how we might arrive at such a formula in the first place. Here is another proof that gives more insight.

First we solve the [homogeneous equation](#)

$$y' + p(t)y = 0.$$

This equation is separable and easy to solve. We do the algebra quickly: The equation can be written as $y' = -p(t)y$. Separating variables gives: $dy/y = -p(t)dt$. Integrating gives:

$\ln(y) = -\int p(t) dt + C$. Now exponentiation gives the general solution to the homogeneous equation:

$$y(t) = Ce^{-\int p(t) dt}$$

To avoid writing integrals repeatedly we let $y_h(t) = e^{-\int p(t) dt}$. So the general homogeneous solution is $y(t) = Cy_h(t)$.

Now consider the [inhomogeneous equation](#)

$$y' + p(t)y = q(t).$$

This is not separable, so we need to do something else. The philosophy behind variation of parameters is to use what we already know. What we know is the homogeneous solution, so we guess that the solution is of the form

$$y(t) = v(t)y_h(t).$$

What we've done is to turn the parameter C in the homogeneous solution into a variable v which depends on t . Hence the name variation of parameters.

Once we've guessed a solution, we substitute it into the inhomogeneous equation to see if we can solve for a $v(t)$ that works. The left-hand side of the inhomogeneous equation is

$$\begin{aligned} y' + p(t)y &= (v(t)y_h(t))' + p(t)v(t)y_h(t) \\ &= v'y_h + vy'_h + pvy_h \\ &= v'y_h + v(y'_h + py_h) \\ &= v'y_h \quad (\text{since } y'_h + py_h = 0). \end{aligned}$$

Equating the left-hand side with the right-hand side we have $v'(t)y_h(t) = q(t)$. This is easy to solve for $v(t)$:

$$v'(t) = q(t)/y_h(t) \Rightarrow v(t) = \int \frac{q(t)}{y_h(t)} dt + C.$$

Now we put this back into our definition of $y(t)$

$$y(t) = v(t)y_h(t) = y_h(t) \left(\int \frac{q(t)}{y_h(t)} dt + C \right) = y_h(t) \int \frac{q(t)}{y_h(t)} dt + Cy_h(t).$$

This is the variation of parameters formula we wanted to derive.

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