ES.1803 Topic 20 Notes

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20 Step and delta functions

20.1 Goals

1. Be able to define the unit step and unit impulse functions and give their properties.
2. Be able to explain why the unit step and unit impulse functions are idealized versions of real physical phenomena.
3. Be able to compute the generalized derivative of a function with jump discontinuities.
4. Be able to compute integrals involving delta functions.
5. Be able to solve DEs with impulses as input.
6. Be able to find the pre and post-initial conditions for a physical model with impulsive input.

20.2 The unit step function

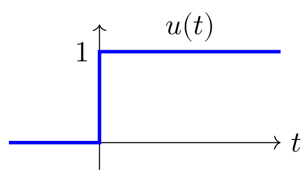
20.2.1 Definition

Let's start with the definition of the unit step function, $u(t)$:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

We do not define $u(t)$ at $t = 0$. Rather, at $t = 0$ we think of it as in transition between 0 and 1.

It is called the unit step function because it takes a unit step at $t = 0$. It is sometimes called the Heaviside function. The graph of $u(t)$ is simple.



We will use $u(t)$ as an idealized model of a natural system that goes from 0 to 1 very quickly. In reality it will make a smooth transition, such as the following.

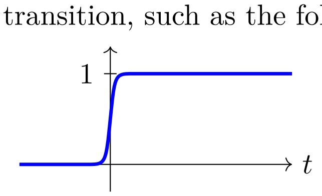


Figure 1. $u(t)$ is an idealized version of this curve

But, if the transition happens on a time scale much smaller than the time scale of the phenomenon we care about, then the function $u(t)$ is a good approximation. It is also much easier to deal with mathematically.

One of our main uses for $u(t)$ will be as a switch. It is clear that multiplying a function $f(t)$ by $u(t)$ gives

$$u(t)f(t) = \begin{cases} 0 & \text{for } t < 0 \\ f(t) & \text{for } t > 0. \end{cases}$$

We say the effect of multiplying by $u(t)$ is that for $t < 0$ the function $f(t)$ is switched off and for $t > 0$ it is switched on.

20.2.2 Integrals of $u'(t)$

From calculus we know that

$$\int u'(t)\,dt = u(t) + c \quad \text{and} \quad \int_a^b u'(t)\,dt = u(b) - u(a).$$

For example:

$$\begin{aligned} \int_{-2}^{5} u'(t)\,dt &= u(5) - u(-2) = 1, \\ \int_{1}^{3} u'(t)\,dt &= u(3) - u(1) = 0, \\ \int_{-5}^{-3} u'(t)\,dt &= u(-3) - u(-5) = 0. \end{aligned}$$

In fact, the following rule for the integral of $u'(t)$ over any interval is obvious

$$\int_a^b u'(t) = \begin{cases} 1 & \text{if 0 is inside the interval } (a, b) \\ 0 & \text{if 0 is outside the interval } [a, b]. \end{cases} \tag{1}$$

Note: If one of the limits is 0, we throw up our hands and refuse to do the integration.

20.2.3 0^- and 0^+

Let 0^- be infinitesimally to the left of 0 and 0^+ infinitesimally to the right of 0. That is,

$$0^- < 0 < 0^+.$$

For a function, $f(0^-)$ is defined as the left hand limit at 0 or, equivalently, the limit from below at 0, provided this limit exists. Likewise, $f(0^+)$ is the right hand limit or the limit from above.

$$f(0^-) = \lim_{t \uparrow 0} f(t) \qquad f(0^+) = \lim_{t \downarrow 0} f(t)$$

Here are some examples of integrals of u' that involve 0^- and 0^+:

$$\int_{-\infty}^{0^+} u'(t)\,dt = 1 \quad (\text{because } -\infty < 0 < 0^+),$$
$$\int_{-\infty}^{0^-} u'(t)\,dt = 0 \quad (\text{because } -\infty < 0^- < 0),$$
$$\int_{0^-}^{0^+} u'(t)\,dt = 1 \quad (\text{because } 0^- < 0 < 0^+).$$

20.3 Preview of generalized functions and derivatives

Of course $u(t)$ is not a continuous function, so, in the 18.01 sense, its derivative at $t = 0$ does not exist. Nonetheless, we saw that we could make sense of the integrals of $u'(t)$. So, rather than throw it away, we call $u'(t)$ the generalized derivative of $u(t)$. You can't do everything with $u'(t)$ you can do with an ordinary function, but we'll see that it can go anywhere we have an input function in 18.03.

20.4 The delta function (unit impulse)

20.4.1 The definition and mathematics of the delta function

Let's delve a little deeper into $u'(t)$. It's clear $u'(t) = 0$ if $t \neq 0$. At $t = 0$ the curve is vertical, so the slope is infinite, i.e., $u'(0) = \infty$. (If you think of $u(t)$ as an idealized version of the curve in Figure 1, then we would say the derivative near 0 gets very large.) We define

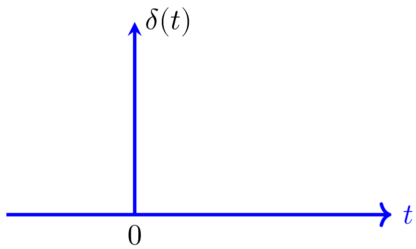
$$\delta(t) = u'(t)$$

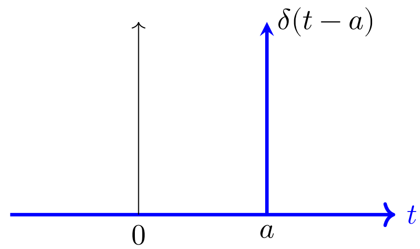
and call it the delta function or the Dirac delta function or the unit impulse function. We have seen the following properties of $\delta(t)$:

1. $\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0. \end{cases}$

2. $\int \delta(t)\,dt = u(t)$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$

Based on Property 1, we 'graph' $\delta(t)$ as an infinite spike at the origin and 0 everywhere else. The integrals show that the 'area' under this graph equals 1 and it is all concentrated at the origin.





We also show $\delta(t-a)$ which is just $\delta(t)$ shifted to the right.

20.5 The non-idealized delta function

Just like the unit step function, the δ function is really an idealized view of nature. In reality, a delta function is nearly a spike near 0, which goes up and down on a time interval much smaller than the scale we are working on. The integral, i.e., area under the curve, is always 1. Its graph might actually look something like

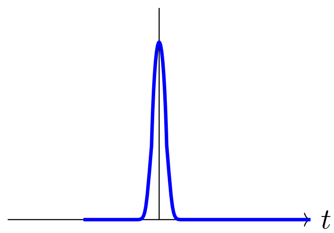


Figure 2. Non-idealized delta function; area under the graph = 1.

The total amount input is still the integral (see Section 20.7 below), or, in geometric terms, the area under the graph. A unit impulse is defined so the area is 1. Later we will consider δ as input to a physical system.

20.6 Delta functions are your friend

20.6.1 Integrals with the delta function

Recall how painful integration could be. In contrast, integrals with delta functions are always easy and involve no techniques of integration.

Suppose we scale $\delta(t)$: the integrals are just scaled.

$$\int_{-5}^{5} 3\delta(t)\,dt = 3, \quad \int_{-5}^{-3} 3\delta(t)\,dt = 0, \quad \int_{0^-}^{0^+} 3\delta(t)\,dt = 3, \quad \int_{0^+}^{\infty} 3\delta(t)\,dt = 0.$$

The integral $\int_a^b f(t)\delta(t)\,dt$ is also easy. If $f(t)$ is continuous at $t=0$ then

$$\int_a^b f(t)\delta(t)\,dt = \begin{cases} f(0) & \text{if } (a,b) \text{ contains } 0 \\ 0 & \text{if } [a,b] \text{ does not contain } 0. \end{cases}$$

That is, integrating against $\delta(t)$ just amounts to evaluating $f(t)$ at $t=0$.

Note 1. If one of the endpoints a or b is 0, the integral cannot be evaluated, so we just throw up our hands and refuse to do it.

Note 2. Technicality: We must have $f(t)$ continuous at $t=0$.

20.6.2 Justification of the formula for integrating with delta functions

We should start by admitting that, in formal mathematic, this is formula is given as the definition of $\delta(t)$, so our arguments will just go to show that it is a reasonable definition. We'll do this in three ways.

Quick reason: $\delta(t)$ is 0 everywhere except $t = 0$, So $f(t)\delta(t)$ is 0 for all $t \neq 0$ and at $t = 0$ it just scales the delta function by $f(0)$. That is, $f(t)\delta(t) = f(0)\delta(t)$.

Reason 1. Since we can interpret the integral as area, we need to show that the 'area' under $f(t)\delta(t)$ is $f(0)$. Figure 2 (above) shows a tall, thin curve near $t = 0$ which approximates $\delta(t)$. Since $f(t)$ is continuous we know that $f(t) \approx f(0)$ near $t = 0$. Thus $f(t)\delta(t)$ is approximated by the graph in the Figure 2 scaled by $f(0)$. Finally, since the area under the curve in Figure 2 is one, if we scale it by $f(0)$ it will have area equal to $f(0)$. As the graph in Figure 2 gets narrower and taller it goes to the graph of $\delta(t)$. As this happens, the approximation we just made will become exact, i.e., as we wanted to show, the area under the $f(t)\delta(t) = f(0)$.

Reason 2. This is a direct argument using integration by parts. First, since $\delta(t) = 0$ for $t \neq 0$ the integral $\int_a^b f(t)\delta(t)\,dt$ must be zero for any interval $[a, b]$ not containing 0. Next, suppose $a < 0 < b$, then we get

$$\begin{aligned}\int_a^b f(t)\delta(t)\,dt &= \int_a^b f(t)u'(t)\,dt \quad \text{(since } \delta = u'\text{)}\\ &= f(t)u(t)\big|_a^b - \int_a^b f'(t)u(t)\,dt \quad \text{(integration by parts)}\end{aligned}$$

Now, since $u(b) = 1$, $u(a) = 0$ and $u(t) = 0$ for $t < 0$ this becomes

$$\begin{aligned}&= f(b) - \int_0^b f'(t)\,dt\\ &= f(b) - f(t)\big|_0^b\\ &= f(b) - f(b) + f(0)\\ &= f(0)\end{aligned}$$

Comparing the first and last expressions in this long sequence of steps, we've shown the result.

Important note: For continuous $f(t)$, the formula

$$f(t)\delta(t) = f(0)\delta(t)$$

is extremely useful. Your life will be much easier if you learn to replace $f(t)\delta(t)$ by $f(0)\delta(t)$.

20.6.3 Shifting by a

If we shift by a, we have $\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t-a) = f(a)$. More generally:

$$\int_c^d f(t)\delta(t-a)\,dt = \begin{cases} f(a) & \text{if } (c, d) \text{ contains a}\\ 0 & \text{if } [c, d] \text{ does not contain a.}\end{cases}$$

Important note: Just as for $\delta(t)$, for continuous $f(t)$ we have, $f(t)\delta(t-a) = f(a)\delta(t-a)$. You should learn to make this replacement.

Example 20.1. (Practice with δ.) Quickly cover up the answers on the right and try to evaluate each of the integrals on the left.

$$\int_{-1}^{3} \delta(t)2e^{4t^2}\,dt = 2, \quad \text{(evaluate } 2e^{4t^2} \text{ at } t = 0)$$

$$\int_{1}^{3} \delta(t)2e^{4t^2}\,dt = 0, \quad \text{(0 is not in [1,3])}$$

$$\int_{0^-}^{3} \delta(t)2e^{4t^2}\,dt = 2, \quad \text{(evaluate } 2e^{4t^2} \text{ at } t = 0)$$

$$\int_{0^-}^{\infty} \delta(t)2e^{-\tan^2(t^3)}\,dt = 2, \quad \text{(evaluate } 2e^{-\tan^2(t^3)} \text{ at } t = 0)$$

$$\int_{-1}^{3} \delta(t-2)2e^{4t^2}\,dt = 2e^{16}, \quad \text{(evaluate } 2e^{2e^{4t^2}} \text{ at } t = 2)$$

$$\int_{3}^{5} \delta(t-2)2e^{4t^2}\,dt = 0, \quad \text{(2 is not in [3,5])}$$

$$\int_{0^-}^{3} \delta(t-2)2e^{4t^2}\,dt = 2e^{16} \quad \text{(evaluate } 2e^{2e^{4t^2}} \text{ at } t = 2),$$

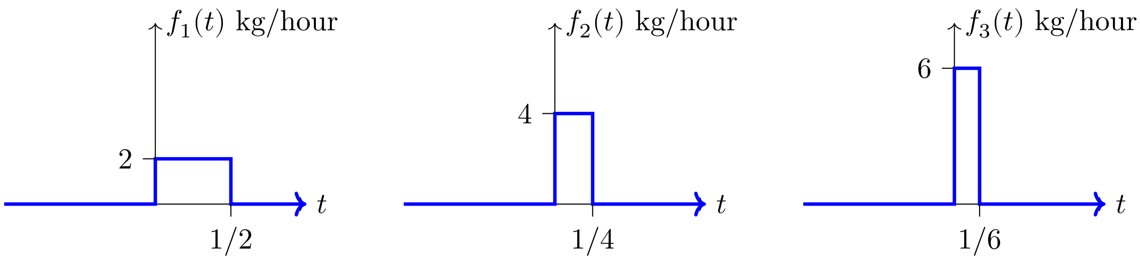
$$\int_{0^-}^{\infty} \delta(t-2)2e^{-\tan^2(t^3)}\,dt = 2e^{-\tan^2(8)} \quad \text{(evaluate } 2e^{-\tan^2(t^3)} \text{ at } t = 2).$$

20.7 The physical interpretation of delta functions as a unit impulse

In general, we will be using δ functions as the input to LTI systems. So, in this subsection, we want to explore what this means. Our goal is to understand what is meant by an impulse and to see that $\delta(t)$ can be thought of as an (idealized) unit impulse.

Example 20.2. Consider the rate equation $\dot{x} + kx = f(t)$. To be specific, assume x is in kilograms of a radioactive substance and t is in hours. This is a rate equation and the derivative $\dot{x}$ and the input $f(t)$ are rates, in units of kg/hour. We then have that the total amount of substance input from time 0^- to time t is $\int_{0^-}^{t} f(\tau)\,d\tau$.

Consider the following possible inputs $f(t)$, shown graphically as box functions.



Look at the input function $f_1(t)$ in the leftmost figure. It is only nonzero in the interval $[0, 1/2]$ during which time it inputs at a constant rate of 2 kg/hour. The total amount input over that time is

$$\int_0^{1/2} f_1(t)\,dt = 1 \text{ kg}.$$

The function f_2 has a higher rate, but acts for a shorter time. The total amount it inputs over time is also 1 kg. The function f_3 is similar: it acts for even a shorter time, but also inputs a total of 1 kg.

If $x(0) = x_0$ kg, then over the interval $[0, 1/2]$ some of the original matter and some of what is added by $f_1(t)$ will decay away. So we'll end with something less than $x_0 + 1$ kg.

Likewise with $f_2(t)$, we add a total of 1 kg over the interval $[0, 1/4]$. Again, there will be decay over the interval, so we'll have less than $x_0 + 1$ at the end of the interval. But, since the interval is shorter, there will be less decay and the amount at the end will be closer to $x_0 + 1$ than with f_1.

If we continue to shorten the time interval in which we input a total of 1 kg, then, in the limiting case, we will dump 1 kg in all at once. In this case, there will be no time for decay and the amount will jump instantaneously from x_0 to $x_0 + 1$, after which it will start decaying. This instantaneous input is called an impulse; an instantaneous input of one unit is called a unit impulse. In a first-order system, an impulse results in an instantaneous jump in the amount of x.

Note, as the time interval gets smaller, the rate needed to add a total of 1 kg must increase. In the limit, when 1 kg is added all at once, the rate must be infinite.

It is worth acknowledging that, in a real physical system, we can't really have an ideal impulse with an infinite rate over an infinitesimal time. But we can come close by having a large rate over a very small time. As long as the time interval is tiny compared to the decay rate, the idealized impulse is a good model. For example, if we add 1 kg of radioactive material in a few seconds, while it decays on a scale of hours, then so little decays while we're adding it, that it is reasonable to model it as an impulse over an infinitesimal time interval.

Claim. Let $u_h(t)$ be the box function of width h and height $1/h$. Then the integral $\int_{-\infty}^{\infty} u_h(t)\,dt = 1$ and

$$\lim_{h \to 0} u_h(t) = \delta(t).$$

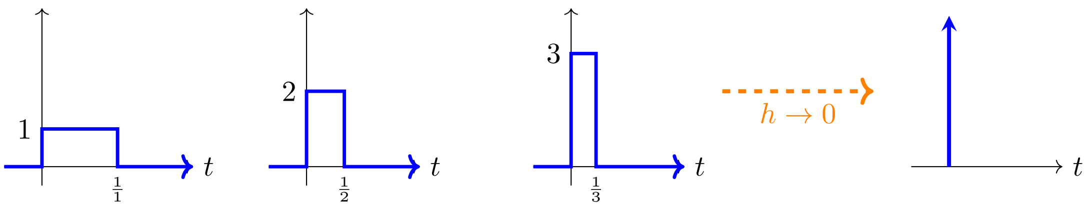
That is, as the boxes get narrower and taller they become the δ function.

Proof. We saw above that $\delta(t)$ was described by two properties

1. $\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0. \end{cases}$

2. $\int \delta(t)\,dt = u(t), \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1.$

The picture below illustrates that $\lim_{h \to 0} u_h(t)$ satisfies property 1. Because all the integrals of $u_h(t) = 1$, the second property is also true of the limit. Because the limit satisfies both properties it must equal $\delta(t)$.



A sequence of box functions $u_h(t)$ limiting to $\delta(t)$.

Summary. Here's a summary of what we've done in this subsection.

1. If $f(t)$ is an input rate. The total amount input over $[a, b]$ is $\int_a^b f(t)\,dt$.
2. A unit impulse adds a total of 1 unit in one instant.
3. If the impulse is at $t = t_0$ then all the input happens at $t = t_0$.
4. We can visualize an impulse as the limit of a sequence of boxes as they get narrower and taller. (Also, look back at the non-idealized delta function in Figure 2: an impulse is the limit of any spike function as it gets narrower and taller.)
5. A unit impulse is modeled by $\delta(t)$.

20.8 Solving DES: pre and post-initial conditions.

The main lesson in this section is that for an nth order equation a delta function, input causes an instantaneous jump in the $(n-1)$st derivative of the output. Once we deal with that, we can use our standard techniques to solve the DE.

Because an impulse causes an instantaneous jump in some value, we have to consider the conditions just before and just after the impulse. Assume the impulse occurs at $t = 0$, then:

At $t = 0^-$, the conditions are pre-initial conditions.

At $t = 0^+$, the conditions are post-initial conditions.

20.8.1 Impulses as input to first-order systems

Example 20.3. Solve $\dot{x} + kx = \delta(t)$ with rest initial conditions.

Solution: This is a first-order exponential decay system. The unit impulse at $t = 0$ causes an instantaneous jump of 1 in the value of x.

On $t < 0$: The DE is *always* $\dot{x} + kx = \delta(t)$. But on this interval $\delta(t) = 0$, so we can simplify the DE to

$$\dot{x} + kx = 0.$$

Since $t < 0$ our initial conditions should use 0^-: $x(0^-) = 0$.

Solving the equation we get: $x(t) = ce^{-kt}$.

Using the initial condition we get: $x(0^-) = c = 0$.

So, $\boxed{\text{on } t < 0,\ x(t) = 0}$. (This should have been obvious to us!)

On $t > 0$: The DE is *always* $\dot{x} + kx = \delta(t)$. But, on this interval $\delta(t) = 0$, so we can simplify the DE to

$$\dot{x} + kx = 0.$$

Since $t > 0$ our initial conditions should use 0^+: The pre-initial condition is $x(0^-) = 0$. The effect of the unit impulse is to cause the value of x to jump by 1 at $t = 0$. That is, $x(0^+) = 1$.

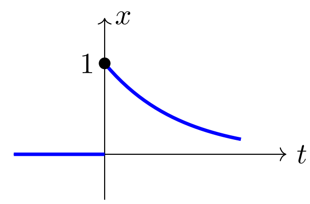
Solving the equation we get: $x(t) = ce^{-kt}$.

Using the initial condition we get: $x(0^+) = c = 1$. So, $\boxed{\text{on } t > 0,\ x(t) = e^{-kt}}$.

The full solution is

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-kt} & \text{for } t > 0. \end{cases}$$

Here is the graph. Note the jump at $t = 0$, followed by exponential decay.



Response from rest to input = $\delta(t)$.

Key: We highlight one key thing to remember in the example above:

In each of the cases $\delta(t) = 0$. That is, when $t < 0$ we have $\delta(t) = 0$. Likewise, when $t > 0$ we have $\delta(t) = 0$.

20.8.2 Impulses as input to second-order systems

Here will give physical reasons for the jump an impulse causes in the first derivative of a second-order system. Later, in Section 20.11, we'll give algebraic reasons for the jump in a system of any order.

Now let's consider the second-order system

$$m\ddot{x} + b\dot{x} + kx = f(t), \tag{2}$$

with input $f(t)$ and output $x(t)$. To be specific, we'll think of this as a spring-mass-damper system with x in meters, t in seconds, and m in kg.

We need to think about the units on $f(t)$. It's clear enough that they are in Newtons, but what are the units of the total input $\int_a^b f(t)\, dt$? Newtons can be written as

$$\text{Newton} = \frac{\text{kg} \cdot \text{ m/sec}}{\text{sec}} = \frac{\text{momentum}}{\text{time}}.$$

That is, force changes momentum over time. We see that the total input has units of momentum.

Following this idea, we see that a unit impulse to this second-order system is a sudden blow, i.e., a large force acting with a short duration, that causes the momentum to jump by one unit.

Example 20.4. Suppose a unit impulse is applied to the system in Equation 2. If the system is at rest before time 0, find the pre- and post-initial conditions.

Solution: Since the system is initially at rest the pre-initial conditions are

$$x(0^-) = 0 \quad \text{and} \quad \dot{x}(0^-) = 0.$$

Since, for this system, the impulse causes a one unit jump in momentum at $t = 0$ we have, at $t = 0^+$, the momentum $m\dot{x}(0^+) = 1$, i.e., the post-initial conditions

$$x(0^+) = 0 \quad \text{and} \quad \dot{x}(0^+) = 1/m.$$

Example 20.5. Assume rest initial conditions and solve the equation

$$2\ddot{x} + 7\dot{x} + 3x = \delta(t).$$

Solution: Following Example 20.4, the post-initial conditions are $x(0^+) = 0$ and $\dot{x}(0^+) = 1/2$. We work on the intervals $t < 0$ and $t > 0$ separately.

On $t < 0$: The input $\delta(t) = 0$, so we have a homogeneous DE with initial conditions

$$2\ddot{x} + 7\dot{x} + 3x = 0, \quad x(0^-) = 0,\ \dot{x}(0^-) = 0.$$

You can easily check that the solution to this is $x(t) = 0$.

So, $\boxed{\text{on } t < 0,\ x(t) = 0.}$

On $t > 0$: The input $\delta(t) = 0$, so we have a homogeneous DE with initial conditions

$$2\ddot{x} + 7\dot{x} + 3x = 0, \quad x(0^+) = 0,\ \dot{x}(0^+) = 1/2.$$

The characteristic roots are $-1/2$ and -3, so

$$x(t) = c_1 e^{-t/2} + c_2 e^{-3t}.$$

Using the initial conditions we find $c_1 = 1/5$ and $c_2 = -1/5$.

So, $\boxed{\text{on } t > 0,\ x(t) = \frac{1}{5}e^{-t/2} - \frac{1}{5}e^{-3t}.}$

The full solution is

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{5}e^{-t/2} - \frac{1}{5}e^{-3t} & \text{for } t > 0. \end{cases}$$

Example 20.6. Solve $4\ddot{x} + x = \delta(t)$ with rest IC.

Solution: The pre-initial conditions are 0, so the post-initial conditions are

$$x(0^+) = 0, \qquad \dot{x}(0^+) = 1/4.$$

On $t < 0$: The differential equation with initial conditions is

$$4\ddot{x} + x = 0; \qquad x(0^-) = 0,\ \dot{x}(0^-) = 0.$$

The solution to this is $x(t) = 0$.

On $t > 0$: The differential equation with initial conditions is

$$4\ddot{x} + x = 0; \qquad x(0^+) = 0,\ \dot{x}(0^+) = 1/4.$$

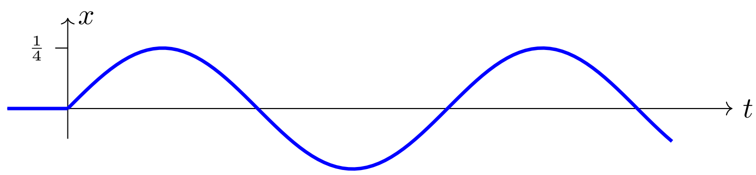
We know the solution to this:

$$x(t) = c_1 \cos(t/2) + c_2 \sin(t/2).$$

We find c_1 and c_2 to match the post-initial conditions: $c_1 = 0$, $c_2 = 1/2$. Therefore, the complete solution is

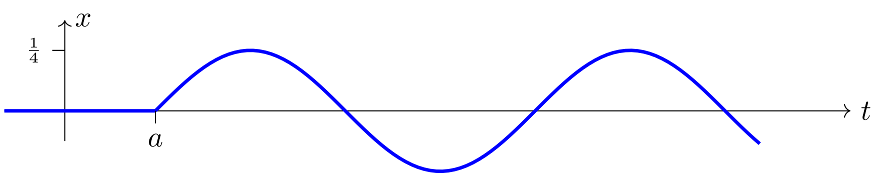
$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2}\sin(t/2) & \text{for } t > 0. \end{cases}$$

Physical explanation. At $t = 0$ an impulse kicks the simple harmonic oscillator into motion. After that, input is 0 and the system is in simple harmonic motion. The jump in momentum corresponds to the corner in graph at 0.



Example 20.7. Solve $4\ddot{x} + x = \delta(t - a)$ with rest IC.

Solution: This is an LTI system, so shifting the input from the previous example a units to the right, shifts the response in the same way.



Example 20.8. (Resonance) Solve the equation $\ddot{x} + x = f(t)$ with rest IC, where the input $f(t)$ is an impulse every 2π seconds of magnitude 3 in the positive direction.

Solution: We have $f(t) = 3\delta(t) + 3\delta(t - 2\pi) + 3\delta(t - 4\pi) + \ldots$. We can solve by solving the DE individually for each input:

$$\ddot{x}_n + x_n = 3\delta(t - 2n\pi)$$

and using superposition. (Note carefully that the rest IC are preserved by superposition. If we did not have rest IC, we would have to be a little more fussy.) The individual equations are exactly like the previous example. We get that the solution to $\ddot{x}_n + x_n = 3\delta(t - 2n\pi)$ is

$$x_n(t) = \begin{cases} 0 & \text{for } t < 2n\pi \\ 3\sin(t - 2n\pi) = 3\sin(t) & \text{for } t > 2n\pi \end{cases}$$

Now, when we superposition these solutions, we see that every 2π seconds we add another copy of $3\sin(t)$ to the output. We call this resonance –the blows come at the natural frequency (every 2π seconds) of the system.

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ 3\sin(t) & \text{for } 0 < t < 2\pi \\ 6\sin(t) & \text{for } 2\pi < t < 4\pi \\ 9\sin(t) & \text{for } 4\pi < t < 6\pi \\ & \dots \end{cases}$$

20.8.3 Impulses as input to third-order systems

Example 20.9. Assume rest initial conditions and solve the equation

$$4(D-1)(D-2)(D-3)x = 4x''' - 24x'' + 44x' - 24x = 5\delta(t).$$

(We give the differential operator in factored form so we can find the characteristic roots easily.)

Solution: For a third-order DE, the jump caused by the impulse follows the same pattern as in the second-order case. That is, the input $5\delta(t)$ causes a jump of 5 in $4x''(t)$ at $t = 0$. Here, the factor of 4 is the coefficient of x''' in the DE. Thus x'' has a jump of $5/4$. The pre-initial conditions are all zero, so after the jump the post-initial conditions are

$$x(0^+) = 0, \quad x'(0^+) = 0, \quad x''(0^+) = 5/4.$$

(In Section 20.11 we will show why this has to be the case.)

On $t < 0$: On this interval, the input $5\delta(t) = 0$. So the differential equation with initial conditions is

$$4(D-1)(D-2)(D-3)x = 0, \quad x(0^-) = 0,\ x'(0^+) = 0,\ x''(0^+) = 0.$$

The solution to this is $x(t) = 0$.

On $t > 0$: We have the homogeneous DE with initial conditions:

$$4(D-1)(D-2)(D-3)x = 0, \quad x(0^+) = 0,\ x'(0^+) = 0,\ x''(0^+) = 5/4.$$

The characteristic roots are 1, 2 and 3, so for $t > 0$ we have

$$x(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{3t}.$$

Using the initial conditions to find the coefficients, we get: $c_1 = \dfrac{5}{8}$, $c_2 = -\dfrac{5}{4}$, $c_3 = \dfrac{5}{8}$.

The full solution is

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{5}{8}e^t - \frac{5}{4}e^{2t} + \frac{5}{8}e^{3t} & \text{for } t > 0. \end{cases}$$

20.9 Box vs. delta as input

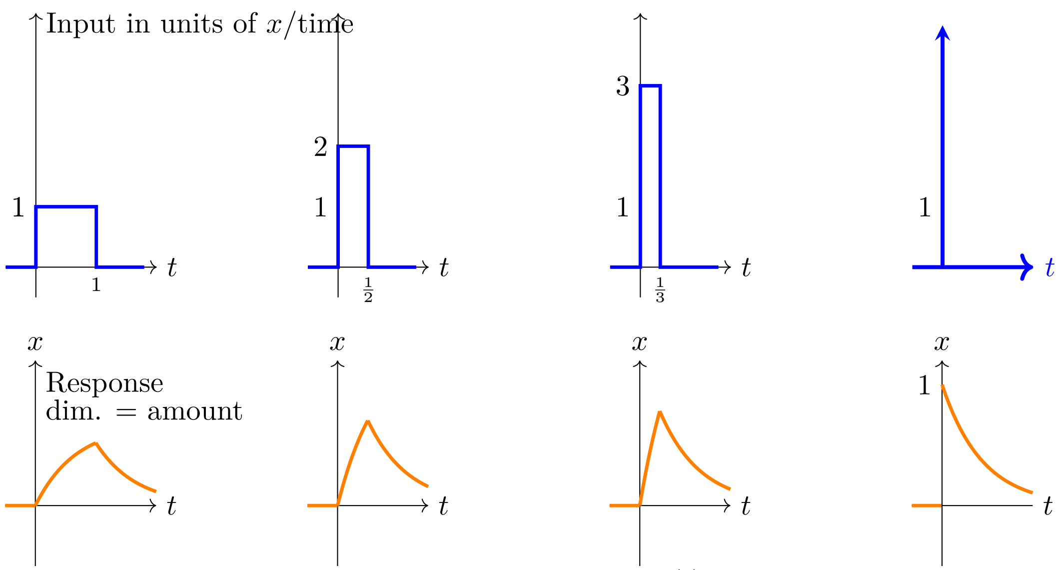
In this section we will compare box functions and delta functions as input. You will see that the delta function is much easier to work with!

Example 20.10. (Box vs. delta.) Let's compare box $u_h(t)$ input with unit impulse ($\delta(t)$) input by solving: $\dot{x} + kx = u_h$ with rest IC.

(*Physical reasoning:*) This models radioactive dumping. u_h is the rate matter is added over time and, as we have seen, the total amount added is $\int_0^h u_h = 1$.

In the figure below the top row of graphs show the input u_h for various values of h. The corresponding responses are shown in the second row of graphs. The total amount input is one, so, since there is decay, at the end of the input interval, we have $x(h) < 1$. After time $t = h$ there is no more input and the response shows exponential decay.

As h goes to 0 the input becomes the unit impulse $\delta(t)$. This is shown in the last graph. Since the input is dumped in all at once the graph jumps from 0 to 1 at $t = 0$. After $t = 0$ the graph is that of exponential decay.



Top: a sequence of box function inputs limiting to $\delta(t)$.
Bottom: response to the sequence of box functions limiting to response to $\delta(t)$.

For completeness we give the exact solution to the IVP $\dot{x} + kx = u_h$ with rest IC.

$$x = \begin{cases} \frac{1}{hk}(1 - e^{-kt}) & \text{for } 0 < t < h \\ \frac{1}{hk}(e^{kh} - 1)e^{-kt} & \text{for } h < t \end{cases}$$

Just as expected, as $h \to 0$ the input becomes δ and the output becomes $x = e^{-kt}$ (i.e., $\lim_{h \to 0} \frac{e^{kh} - 1}{hk} = 1$)

20.10 Generalized derivatives

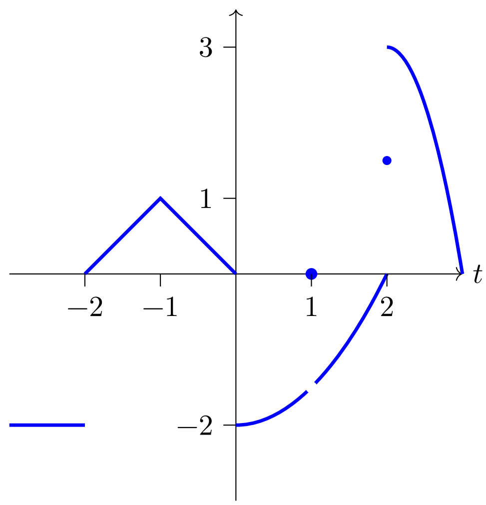
So far we have only one generalized derivative: $\dot{u}(t) = \delta(t)$. In this section we will learn to compute them for any function with jump discontinuities.

Definition. We say a function $f(t)$ has a jump discontinuity at $t = t_0$ if its graph is continuous on both the left and right, and there is a jump at t_0.

Formally this means that both left and right limits $\lim_{t \uparrow t_0^-} f(t)$ and $\lim_{t \downarrow t_0^+} f(t)$ exist, but are different. The jump at t_0 is defined as the difference

$$\lim_{t \to t_0^+} f(t) - \lim_{t \to t_0^-} f(t)$$

Example 20.11. The graph of a function $f(t)$ is shown below. It has jump discontinuities at -2, 0 and 2. The jumps are respectively 2, -2 and 3. The graph also has a corner at -1. That is, the graph is continuous at $t = -1$, but the derivative has a jump there.



Notes. 1. Not all discontinuities result in jumps. At $t = 1$ the jump between the left and right limits is 0. You could say the function jumps from -1.5 to 0 and back to -1.5 for a net jump of 0.

2. The value of $f(2)$ (represented by a dot on the graph) did not play a role in the value of the jump at $t = 2$. The jump is the size of gap between the left and right branches of the curve. You could say the function jumps from 0 to 1.5 to 3 for a net jump of 3.

3. At $t = 0$ the jump is negative because the right branch of the graph is below the left branch.

Generalized derivative: If a function is smooth except for some jump discontinuities and corners then its generalized derivative is:

- the regular derivative away from the jumps and corners.
- delta functions at the jumps. The coefficient on the delta function is the size of the jump.
- undefined at the corners.

Reason. Just as with the unit step function, the graph has 'infinite' slope at a jump and the integral of the derivative should give the original function. This is exactly what δ functions do at jumps.

Example 20.12. Suppose

$$f(t) = \begin{cases} -2 & \text{for } t < -2 \\ t+2 & \text{for } -2 < t < -1 \\ -t & \text{for } -1 < t < 0 \\ t^2/2 - 2 & \text{for } 0 < t < 2 \\ 3 - 3(t-2)^2 & \text{for } 2 < t. \end{cases}$$

Find the generalized derivative $f'(t)$.

Solution: We just take the regular derivative and add delta functions at the jump discontinuities. Note that the corner when $t = -1$ becomes a jump in the derivative.

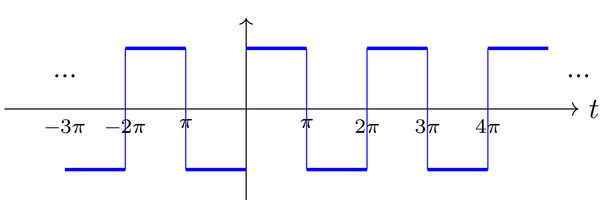
$$f'(t) = \underbrace{2\delta(t+2) - 2\delta(t) + 3\delta(t-3)}_{\text{singular part}} + \underbrace{\begin{cases} 0 & \text{for } t < -2 \\ 1 & \text{for } -2 < t < -1 \\ -1 & \text{for } -1 < t < 0 \\ t & \text{for } 0 < t < 2 \\ -6(t-2) & \text{for } 2 < t. \end{cases}}_{\text{regular part}}$$

Vocabulary: We name the two parts of the generalized derivative. The part which is the regular derivative is called the regular part and the part with delta functions due to the jumps is called the singular part. These are labeled in the example above.

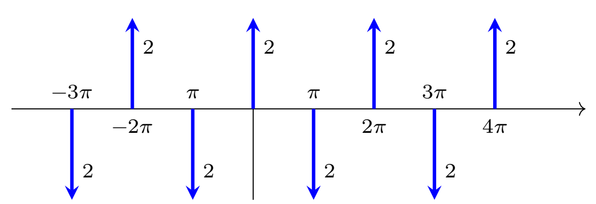
Example 20.13. Derivative of a square wave

The graphs below are of a function $\text{sq}(t)$ (called a square wave) and its derivative. The function alternates every π seconds between ± 1. The derivative $\text{sq}'(t)$ is clearly 0 everywhere except at the jumps. A jump of $+2$ gives a (generalized) derivative of 2δ and a jump of -2 gives a (generalized) derivative of -2δ. Thus we have

$$\text{sq}'(t) = \ldots + 2\delta(t+2\pi) - 2\delta(t+\pi) + 2\delta(t) - 2\delta(t-\pi) + 2\delta(t-2\pi) - 2\delta(t-3\pi) + \ldots$$



Graph of $\text{sq}(t)$ = square wave



Graph of $\text{sq}'(t)$ = impulse train

Note that we put the weight of each delta function next to it. We use the convention that $-2\delta(t)$ is represented by a downward arrow with the weight 2 next to it. That is, the sign is represented by the direction of the arrow, so the weight is positive.

20.11 Generalized derivative: checking solutions, explanation for jumps in post-initial conditions

In this section we will check the answers to a few of our previous examples by plugging them into the original DE. This should give you a feel for how a delta function as input causes a jump in the $(n-1)$st derivative of an nth-order equation.

Example 20.14. (Check the solution in Example 20.3)

The DE $\dot{x} + kx = \delta(t)$ has solution $x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-kt} & \text{for } t > 0. \end{cases}$.

This has a jump of 1 at $t = 0$, so $\dot{x}(t)$ is a generalized derivative:

$$\dot{x}(t) = \delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ -ke^{-kt} & \text{for } t > 0 \end{cases}$$

We now check:

$$\dot{x} + kx = \left(\delta + \begin{cases} 0 & \text{for } t < 0 \\ -ke^{-kt} & \text{for } t > 0 \end{cases}\right) + k\left(\begin{cases} 0 & \text{for } t < 0 \\ e^{-kt} & \text{for } t > 0 \end{cases}\right) = \delta(t).$$

Notice that the jump in x yielded a delta function in $\dot{x}$.

Example 20.15. (Check Example 20.5) Here the DE was $2\ddot{x} + 7\dot{x} + 3x = \delta(t)$ and the solution was

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{5}e^{-t/2} - \frac{1}{5}e^{-3t} & \text{for } t > 0. \end{cases}$$

$x(t)$ has no jump at $t = 0$, so it has a regular derivative

$$\dot{x}(t) = \begin{cases} 0 & \text{for } t < 0 \\ -\frac{1}{10}e^{-t/2} + \frac{3}{5}e^{-3t} & \text{for } t > 0. \end{cases}$$

Since $\dot{x}(t)$ has a jump of 1/2 at $t = 0$, we will get a δ function in $\ddot{x}(t)$:

$$\ddot{x}(t) = \frac{1}{2}\delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{20}e^{-t/2} - \frac{9}{5}e^{-3t} & \text{for } t > 0. \end{cases}$$

It is now easy to check that $2\ddot{x} + 7\dot{x} + 3x = \delta(t)$.

In particular, note that $2\ddot{x}(t) = \delta(t) +$ regular part. This explains why, in Example 20.5 we wanted $\dot{x}$ to jump by 1/2, i.e., then $\ddot{x}$ had singular part $\delta(t)/2$, so $2\ddot{x}$ had singular part $\delta(t)$, which is needed for the left hand side of the DE to equal $\delta(t)$.

Example 20.16. (Check Example 20.9) We will do this check more quickly than the previous two. Also, we will leave out the case $t < 0$ since it is always 0. As we do the computation, notice that $x(0^-) = x(0^+)$ and $x'(0^-) = x'(0^+)$, so there is no jump until $x''(0^-) = 0$ and $x''(0^+) = 5/4$. Thus $\delta(t)$ appears in $x'''(t)$ and the jump is such that $4x'''(t) = 5\delta(t) +$ regular part.

To check the solution, we compute each term in the DE:

$$\begin{aligned}
-24x &= -24\left(\frac{5}{8}e^t - \frac{5}{4}e^{2t} + \frac{5}{8}e^{3t}\right)\\
44x' &= 44\left(\frac{5}{8}e^t - \frac{5}{2}e^{2t} + \frac{15}{8}e^{3t}\right)\\
-24x'' &= -24\left(\frac{5}{8}e^t - 5e^{2t} + \frac{45}{8}e^{3t}\right)\\
4x''' &= 4\left(\frac{5}{8}e^t - 10e^{2t} + \frac{135}{8}e^{3t} + \frac{5}{4}\delta(t)\right)
\end{aligned}$$

Adding this up verifies that $x(t)$ is a solution to the DE: $4x''' - 24x'' + 44x' - 24x = 5\delta(t)$.