ES.1803 Topic 21 Notes

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21 Fourier series introduction

21.1 Goals

1. Know the definition and terminology of Fourier series.
2. Know how to compute the Fourier coefficients for a given periodic function.

21.2 Introduction

So far in 18.03 we have spent a lot of time solving the constant coefficient differential equation $P(D)x = f(t)$, where $f(t)$ is sinusoidal. Our main goal in the next few topics is to extend this to solve $P(D)x = f(t)$, where $f(t)$ can be any periodic function. The outline of our plan is fairly simple:

1. Fourier series. If $f(t)$ is periodic, we'll see that we can write it as a superposition of sine and cosine functions. For example,

$$f(t) = 1 + \cos(t)/2 + \cos(2t)/4 + \cos(3t)/8 + ...$$

2. Linearity. Then we can use the sinusoidal response formula for each term and use the superposition principle to solve $P(D)x = f(t)$.

21.3 Terminology

21.3.1 Frequency terminology

Angular frequency or circular frequency is in radians/sec or more generally in radians/time.

Frequency is in cycles/time –often in cycles/sec = hertz.

Example 21.1. Consider the function $\cos(3t)$, where t is in seconds.
The angular frequency is $\omega = 3\,\text{rad/sec}$.
The frequency is $f = \omega/(2\pi) = 3/(2\pi)\,\text{hz}$.
The period is $T = 1/f = 2\pi/\omega = 2\pi/3\,\text{sec}$.

Depending on the context, we will use frequency or angular frequency. To make matters messier we will often say frequency when we mean angular frequency.

21.3.2 Fourier series terminology

Here we give an example Fourier series. We'll use it to define the terminology we'll be using.

Example 21.2. Suppose we have

$$f(t) = \frac{3}{2} + \cos(\pi t) + \frac{\cos(2\pi)t}{2} + \frac{\cos(3\pi t)}{3} + ... + \sin(\pi t) + \frac{\sin(2\pi t)}{2^2} + \frac{\sin(3\pi t)}{3^2} + ...$$

This is a Fourier series. It has the following properties.

1. A Fourier series is sum of sines and cosines. All of the terms have a common period. In this example, every term has period 2. (Most terms also have a smaller period.)
2. It has a base angular frequency also called the fundamental angular frequency. In this example, the base frequency is π.
3. The frequency in each term is a multiple of the base frequency.
4. The base period corresponds to the base frequency. In this example, the base period is 2.
5. The Fourier coefficients are the coefficients of the sine and cosine terms. In this example, the cosine coefficients are
$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$
and the sine coefficients are
$$1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \ldots$$
The constant term is $\frac{3}{2}$. This is called the DC term. DC stands for direct current.

21.4 Periodic functions

Definition. $f(t)$ is periodic with period $p > 0$ if $f(t+p) = f(t)$ for all t.

Examples: $\cos(t)$ has period 2π, $\cos(3t)$ has period $2\pi/3$, $\cos\left(\frac{\pi}{L}t\right)$ has period $2L$

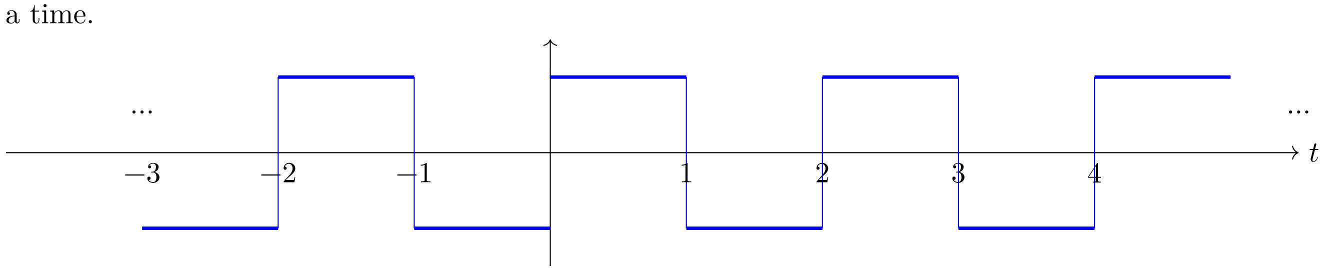
Important point. Just like a complex number has multiple arguments, a periodic function has multiple periods. For example, $\cos(t)$ has periods 2π, 4π, 6π, ...

An even more extreme case is the constant function $f(t) = 1$ which has period p for any $p > 0$.

Specifying a periodic function. For a periodic function, it's enough to specify two things:
1. The period
2. The values of the function over 1 period.

Example 21.3. Period 2 square wave: period $= 2$, over one period $f(t) = \begin{cases} -1 & \text{for } -1 < t < 0 \\ 1 & \text{for } 0 < t < 1 \end{cases}$

We can now plot the $f(t)$ by plotting the period given and then shifting that one period at a time.



Graph of $f(t)$ = period 2 square wave

21.5 Fourier's theorem

Theorem (Fourier): Suppose $f(t)$ has period $p = 2L$ then

1. We can write $f(t)$ as a Fourier series

$$\begin{aligned} f(t) \sim \; & \frac{a_0}{2} + a_1 \cos\left(\frac{\pi}{L}t\right) + a_2 \cos\left(2\frac{\pi}{L}t\right) + a_3 \cos\left(3\frac{\pi}{L}t\right) + \cdots \\ & + b_1 \sin\left(\frac{\pi}{L}t\right) + b_2 \sin\left(2\frac{\pi}{L}t\right) + b_3 \sin\left(3\frac{\pi}{L}t\right) + \cdots \\ = \; & \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right) \end{aligned}$$

(We use $\sim$ instead of an equal sign because the two sides might differ for a few values of t. From now on, we will simply use an equal sign and ignore this fact.)

2. The base period of $f(t)$ is $2L$. Each term in the Fourier series has period $2L$. For example, $\cos\left(3\frac{\pi}{L}t\right)$ has period $2L/3$, but also $4L/3$, $\boxed{6L/3 = 2L}$

3. The series has base angular frequency $= \omega = \dfrac{\pi}{L}$. Every term in the series has an angular frequency which is a multiple of $\frac{\pi}{L}$.

4. The Fourier coefficients are given by:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}t\right)\,dt \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(n\frac{\pi}{L}t\right)\,dt \end{aligned}$$

Notes

1. All Fourier terms $a_n \cos\left(n\frac{\pi}{L}t\right)$, $b_n \sin\left(n\frac{\pi}{L}t\right)$ have period $2L$.

2. Accept the formulas for Fourier coefficients for now, we will give more explanation later.

3. The integrals must be over one full period. We showed them from $-L$ to L because that is the interval of integration we use most often. Sometimes, the function is defined in a way that makes the interval from 0 to $2L$ a better choice.

Example 21.4. Compute the Fourier series for the square wave of period $= 2\pi$,

$$\text{sq}(t) = \begin{cases} -1 & \text{for } -\pi < t < 0 \\ 1 & \text{for } 0 < t < \pi. \end{cases}.$$

Solution: The half period $L = \pi$. So, for $n \neq 0$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \int_{-\pi}^{0} -\cos(nt)\,dt + \int_{0}^{\pi} \cos(nt)\,dt = -\frac{\sin(nt)}{n\pi}\bigg|_{-\pi}^{0} + \frac{\sin(nt)}{n\pi}\bigg|_{0}^{\pi} = 0.$$

For $n = 0$, $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\,dt = 0$. (Always compute a_0 separately.)

Likewise

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin(nt)\,dt \\
&= \left.\frac{\cos(nt)}{n\pi}\right|_{-\pi}^{0} - \left.\frac{\cos(nt)}{n\pi}\right|_{0}^{\pi} \\
&= \frac{1-\cos(-n\pi)}{n\pi} - \frac{\cos(n\pi)-1}{n\pi} = \frac{2}{n\pi}(1-\cos(n\pi)) = \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}.
\end{aligned}$$

Thus, $f(t) = \dfrac{4}{\pi}\left(\sin t + \dfrac{1}{3}\sin 3t + \dfrac{1}{5}\sin 5t + \cdots\right)$.

21.5.1 Simple formulas for certain angles

Here are several formulas you know and should always use:

1. $\cos(n\pi) = (-1)^n$ *always make this substitution*

2. $\sin(n\pi) = 0$ *always make this substitution*

The values $\sin\left(\frac{n\pi}{2}\right)$ and $\cos\left(\frac{n\pi}{2}\right)$ do not have an easier formula. For example for $n = 0, 1, 2, 3, 4$ we have $\sin\left(\frac{n\pi}{2}\right) = 0, 1, 0, -1, 0$.