ES.1803 Topic 22 Notes

Jeremy Orloff

22 Fourier series introduction: continued

22.1 Goals

1. Be able to compute the Fourier coefficients of even or odd periodic function using simplified formulas.
2. Be able to determine the decay rate of the coefficients of a Fourier series.
3. Be able to predict the decay rate of the Fourier coefficients based on the smoothness of the original function.

22.2 Introduction

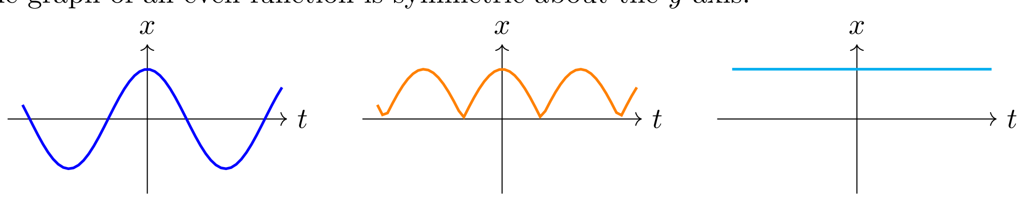
In this topic we continue our introduction to Fourier series. We start by looking at some tricks for computing Fourier coefficients. Then we will talk about more conceptual notions, including the convergence properties of Fourier series and the decay rate of Fourier coefficients. At the end, we will look at the orthogonality relations which explain the formulas for Fourier coefficients.

22.3 Calculation tricks: even and odd functions

22.3.1 Even and odd functions

A function is an even function if $f(-t) = f(t)$ for all t.

- The graph of an even function is symmetric about the y-axis.



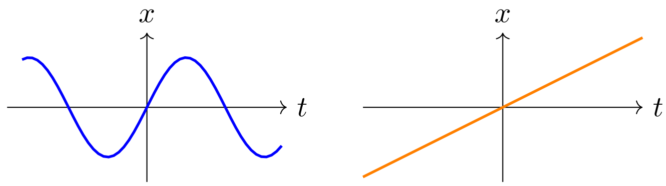
Graphs of some even functions

- Examples of even functions: 1, t^2, t^4, ..., $\cos(\omega t)$. In general, even functions are built out of even powers of t. Note that, the power series for $\cos(\omega t)$ has only even powers.
- By symmetry we have the following key integration fact for even functions:

$$\int_{-L}^{L} f(t)\,dt = 2\int_{0}^{L} f(t)\,dt \quad \text{for any even } f(t).$$

A function is an odd function if $f(-t) = -f(t)$ for all t.

- The graph of an odd function is symmetric about the origin.



Graphs of some odd functions

- Examples of odd functions: t, t^3, t^5, ..., $\sin(\omega t)$. In general, odd functions are built out of odd powers of t. Note that, the power series for $\sin(\omega t)$ has only odd powers.
- By symmetry we have the following key integration fact for odd functions:

$$\int_{-L}^{L} f(t)\,dt = 0 \quad \text{for any odd } f(t).$$

Products of even and odd functions

We give the rules in a kind of short-hand. You can remember these rules by thinking about powers of t, e.g., $t^4 \cdot t^7 = t^{11}$, so even · odd is odd.

- even · even = even, e.g., $t^4 \cdot t^6 = t^{10}$
- odd · odd = even, e.g., $t^3 \cdot t^5 = t^8$
- odd · even = odd, e.g., $t^3 \cdot t^6 = t^9$

22.3.2 Fourier coefficients of even and odd functions

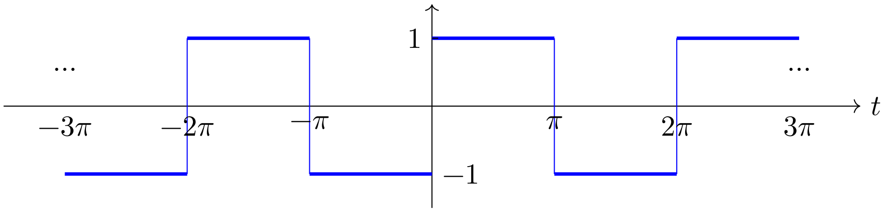
- If $f(t)$ is even, then $b_n = 0$ and $a_n = \frac{2}{L}\int_0^L f(t)\cos(\frac{n\pi}{L}t)\,dt$.
- If $f(t)$ is odd, then $a_n = 0$, and $b_n = \frac{2}{L}\int_0^L f(t)\sin(\frac{n\pi}{L}t)\,dt$.

Reason. Assume $f(t)$ is even. Then the multiplication rules for even functions imply $f(t)\cos(\omega t)$ is even. So, $a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt.$

Likewise, the rules imply $f(t)\sin(\omega t)$ is odd. So, $b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)\,dt = 0.$

The argument is similar when $f(t)$ is odd.

Example 22.1. In the previous topic notes we met the period 2π square wave, which over one period has the formula $\text{sq}(t) = \begin{cases} -1 & \text{for } -\pi < t < 0 \\ 1 & \text{for } \ 0 < t < \pi. \end{cases}$



Graph of sq(t) = square wave

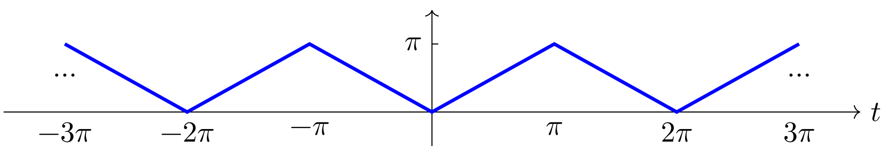
Since the period is 2π, we have $L = \pi$. Since sq(t) is odd, we know that $a_n = 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \mathrm{sq}(t)\sin(nt)\,dt = \frac{2}{\pi}\int_0^{\pi}\sin(nt)\,dt = -\frac{2}{n\pi}\cos(nt)\Big|_0^{\pi} = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

We have found the Fourier series for sq(t):

$$\mathrm{sq}(t) = \sum_{n=1}^{\infty} b_n \sin(nt) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}.$$

Example 22.2. Triangle wave function (also called the continuous sawtooth function). Let $f(t)$ have period 2π and $f(t) = |t|$ for $-\pi \le t \le \pi$. Compute the Fourier series of $f(t)$.



Graph of $f(t)$ = triangle wave

Since $f(t)$ is an even function, we know that $b_n = 0$ and for $n \neq 0$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt \\ &= \frac{2}{\pi}\left[\frac{t\sin(nt)}{n} + \frac{\cos(nt)}{n^2}\right]_0^{\pi} = \frac{2}{n^2\pi}((-1)^n - 1) = \begin{cases} -\frac{4}{n^2\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

As usual, we compute a_0 separately: $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |t|\,dx = \dfrac{2}{\pi}\int_0^{\pi} t\,dt = \pi$.

Thus we have the Fourier series for $f(t)$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}.$$

22.4 Summing Fourier series

We can use the sum of a finite number of terms from a Fourier series to approximate the original function. The applet `https://web.mit.edu/jorloff/www/OCW-ES1803/fourierapproximation.html` illustrates this. In the following sections we will bring out the following key points:

- The first few terms of the Fourier series approximate the shape of the function, not necessarily the value of the function at any one point.
- At points of continuity, the Fourier series converges to the original function.
- The smoother the function, the faster the Fourier series converges to the function.
- At jumps in the graph, no matter how many terms you use, the Fourier series always overshoots the graph near that point.

22.5 Convergence of Fourier series

Piecewise smooth: The period $2L$ function $f(t)$ is called piecewise smooth if there are only a finite number of points $0 \leq t_1 < t_2 < \ldots < t_n \leq 2L$ where $f(t)$ is not differentiable and at each of these points the left and righthand limits

$$f(t_i^+) = \lim_{t \to t_i^+} f(t) \quad \text{and} \quad f(t_i^-) = \lim_{t \to t_i^-} f(t)$$

exist (although they might not be equal).

In short, a function is piecewise smooth if it is smooth except at a discrete set of points where is has jump discontinuities.

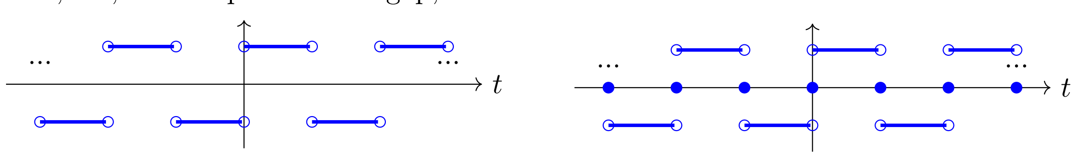
Here is our main theorem about convergence of Fourier series. We will not prove it in ES.1803.

Theorem: If $f(t)$ is piecewise smooth and periodic, then the Fourier series for f:

1. Converges to $f(t)$ at values of t where f is continuous.

2. Converges to the average of $f(t^-)$ and $f(t^+)$ at values of t where $f(t)$ has a jump discontinuity.

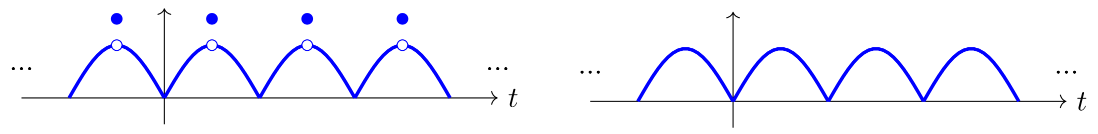
Example 22.3. Square wave. The square wave in the example above has jump discontinuities. No matter how we specify the endpoint behavior of $\text{sq}(t)$, the Fourier series converge to 0, i.e., the midpoint of the gap, at the discontinuities.



Original $\text{sq}(t)$ Fourier series

Example 22.4. The triangle wave in the example above is continuous so its Fourier series converges to the original function $f(t)$.

Example 22.5. We give one more graphical example. Here the original function has discontinuities –admittedly somewhat artificial. Since the left and righthand limits are the same at each discontinuity the Fourier series is continuous.



Original $f(t)$ Fourier series

22.5.1 Decay rate of Fourier coefficients

Sequences like $a_n = 1/n$ and $b_n = 1/n^2$ go to 0 as n goes to infinity. We say they decay to 0. Clearly b_n goes to 0 faster than a_n. We will say 'b_n decays like $1/n^2$'.

In general we will ignore constant factors, so, for example, we say $4/(n\pi)$ decays like $1/n$.

Example 22.6. The Fourier coefficients of $\text{sq}(t)$ are

$$a_n = 0 \quad \text{and} \quad b_n = \begin{cases} 4/(n\pi) & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

We say these coefficients decay like $1/n$.

Example 22.7. The triangle wave looked at above has Fourier coefficients

$$b_n = 0 \quad \text{and} \quad a_n = \begin{cases} -4/(n^2\pi) & \text{for } n \text{ odd} \\ 0 & \text{for even } n \neq 0. \end{cases}$$

So these coefficients decay like $1/n^2$.

Example 22.8. The coefficients $a_n = 1/(n + n^2)$ decay like $1/n^2$.

Example 22.9. If a Fourier series has $a_n = 1/n$ and $b_n = 1/n^2$, we say a_n decays like $1/n$ and b_n decays like $1/n^2$. The Fourier coefficients as a whole decay like the slower of the two rates. That is, they decay like $1/n$.

Example 22.10. The function $f(t) = 3\cos(t) + 5\cos(2t)$ is a finite Fourier series. The coefficients are $a_0 = 0$, $a_1 = 3$, $a_2 = 5$, $a_3 = 0$, $a_4 = 0$, ... We say these coefficients decay like 0.

22.5.2 Important heuristics

- If a function has a jump discontinuity, then its Fourier coefficients decay like $\frac{1}{n}$, e.g., the square wave.
- If a function has a corner, then its Fourier coefficients decay like $\frac{1}{n^2}$, e.g., the triangle wave
- A smooth function has Fourier coefficients that decay like $\frac{1}{n^3}$ or faster.
- The smoother the function, the faster the coefficients decay.

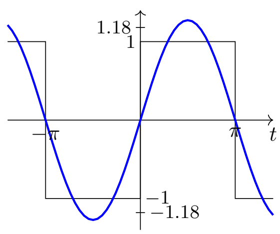
22.6 Gibbs' phenomenon

22.6.1 Non-local nature of Fourier series

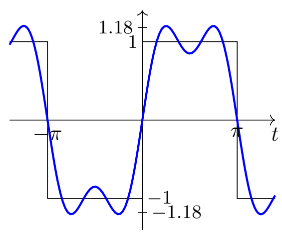
Generally speaking, if we sum the first few terms of a Fourier series, it will match the overall shape of the original function. An analogy is the way a squares fit of data points matches the shape of the data without necessarily going through any of the data points.

The figures below show the square wave and its Fourier series summed to some number of terms. The first plot uses just the first term, i.e., $\frac{4}{\pi}\sin(t)$. Notice how it matches the general oscillation of the square wave without matching it well at any particular place.

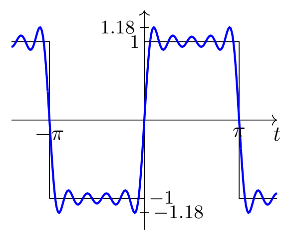
The second plot uses the terms out to $n = 3$, i.e., $\frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3}\right)$. This fits the square wave a little better than the first plot. The third plot uses the terms out to $n = 21$. This fits the square even better.



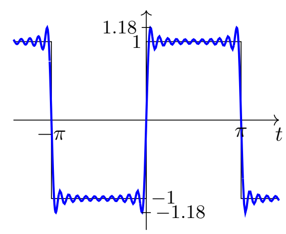
Sum up to $n = 1$



Sum up to $n = 3$



Sum up to $n = 9$



Sum up to $n = 21$

22.6.2 Gibbs' phenomenon

In the figures above, notice that the peak of the reconstructed square wave always overshoot the square 0.18, i.e., it goes up to about 1.18 or down to -1.18. As the number of terms increases, the point where the overshoot occurs moves closer to the point of discontinuity, but never disappears.

This is a general phenomenon, called Gibbs' phenomenon. For any periodic function with a jump discontinuity, summing any number of terms from its Fourier series will *always* overshoot the jump by about 9% of the size of the jump. For example, the square wave has a jump of size 2, so the overshoot is about $2 \cdot 0.09 = 0.18$. Gibbs' phenomenon is extremely important in many applications, e.g., digital filtering of signals.

We won't prove Gibbs' phenomenon in ES.1803. For those who are interested, we've posted an enrichment note with the proof. It should accessible to anyone who knows calculus.

The applet
`https://web.mit.edu/jorloff/www/OCW-ES1803/fourierapproximation.html`
shows this overshoot in several cases.

22.7 Orthogonality relations

22.7.1 Orthognality relation integrals

The key to the integral formulas for Fourier coefficients are the orthogonality relations. These are the following integral formulas that say certain trigonometric integrals are either 0 or 1.

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)\,dt = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \\ 2 & n = m = 0 \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)\,dt = 0$$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi}{L}t\right)\sin\left(\frac{m\pi}{L}t\right)\,dt = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

Proof. We have two methods to do this. We will carry out the first, but only mention the second.

Method 1: Use the following trigonometric identities

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$

Method 2: Use $\cos(at) = \frac{e^{iat}+e^{-iat}}{2}$ etc.

Using method 1 we get the following if $n \neq m$:

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)\,dt &= \frac{1}{L}\int_{-L}^{L} \frac{\cos\left(\frac{(n+m)\pi}{L}t\right)+\cos\left(\frac{(n-m)\pi}{L}t\right)}{2}\,dt \\ &= \frac{1}{2L}\left[\frac{\sin\left(\frac{(n+m)\pi}{L}t\right)}{(n+m)\pi/L} + \frac{\sin\left(\frac{(n-m)\pi}{L}t\right)}{(n-m)\pi/L}\right|_{-L}^{L} \\ &= 0.\end{aligned}$$

The last equality is easy to see since every term is 0 when $t = \pm L$.

The case $n = m$ is special because then $n - m = 0$. It is easy to use the first trig identity above with $\alpha = \beta$, i.e., $\cos(\alpha)\cos(\alpha) = (\cos(2\alpha)+1)/2$, to see that the integral in this case is 1. All the other orthogonality relations are proved in a similar fashion.

The term orthogonality comes from linear algebra, where we say two vectors are orthogonal if there dot product is 0. It turns out that we can think of $\int_{-L}^{L} f(t)g(t)\,dt$ as a dot product (usually called inner product) between f and g. So the orthogonality relations say that, for $n \neq m$, the functions $\cos(n\pi t/L)$ and $\cos(m\pi t/L)$ are orthogonal.

22.7.2 Using orthogonality relations to show the formula for Fourier coefficients

The orthogonality relations allow us to see that if $f(t)$ is written as a Fourier series, then the coefficients must be given by the integral formulas we've been using.

So suppose $f(t)$ has Fourier series':

$$f(t) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi}{L}t\right) + a_2 \cos\left(\frac{2\pi}{L}t\right) + \cdots + b_1 \sin\left(\frac{\pi}{L}t\right) + b_2 \sin\left(\frac{2\pi}{L}t\right) + \cdots$$

Then for $n > 0$

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt = \frac{1}{L}\int_{-L}^{L} \Big[&\frac{a_0}{2}\cos\left(\frac{n\pi}{L}t\right) + a_1\cos\left(\frac{\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right) \\ &+ a_2\cos\left(\frac{2\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right) + \cdots \\ &+ b_1\sin\left(\frac{\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right) + b_2\sin\left(\frac{2\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right) + \cdots\Big]\,dt\end{aligned}$$

Now we can apply the orthogonality relations to each term. All of them are 0, except the term with $a_n \cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right)$ which, again by the orthogonality relations, integrates to a_n. Thus, $\displaystyle\frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi}{L}t)\,dt = a_n$. Which is exactly the formula for the Fourier coefficient. The formulas for a_0 and b_n are found in the same way.

22.8 Hearing a musical triad: C-E-G

Here is a simplified Fourier-centric view of how humans hear sound.

Sound reaches your ear as a pressure wave. For example

$$f(t) = a_1\cos(\omega_1 t) + a_2\cos(w_2 t) + \cdots$$

Do the ears do Fourier analysis?

Answer: Yes! The ear contains hair-like structures called stereocilia. These are different sizes and, so, resonate at different frequencies. As they vibrate they stimulate nerves, which then send signals to the brain. Thus, for each frequency in the pressure wave, the brain is getting a signal from the nerves attached to the stereocilia which vibrate at that frequency. The greater the amplitude in the input wave the greater the amplitude of the signal sent to the brain.

Does the brain do Fourier synthesis?

Answer: Yes! It is up to the brain to combine all the nerve signals at different frequencies into a single signal which it then interprets.