ES.1803 Topic 23 Notes

Jeremy Orloff

23 Fourier sine and cosine series; calculation tricks

23.1 Goals

1. Be able to use various calculation shortcuts for computing Fourier series:
   shifting and scaling $f(t)$
   shifting and scaling t
   differentiating and integrating known series.

2. Be able to find the sine and cosine series for a function defined on the interval $[0, L]$

3. Understand the distinction between $f(x)$ defined on $[0, L]$ and it's even and odd periodic extensions.

23.2 Introduction

This topic is split into two subtopics. First, we look at a few more calculation tricks. The common idea in these tricks is to use the Fourier series of one function to find the Fourier series of another. A simple example is if we scale a function, say $g(t) = 5f(t)$, the Fourier series for $g(t)$ is 5 times the Fourier series of $f(t)$.

Next, we'll look at functions $f(x)$ that are only defined on the interval $[0, L]$. This is in preparation for our later study of the heat and wave equations. This function is not periodic –it's not even defined for all x. By extending $f(x)$ to an even or odd periodic function we can write the original function $f(x)$ as a sum of sines (sine series) or a sum of cosines (cosine series).

23.3 Calculation shortcuts

One of our goals is to avoid computing integrals when finding the Fourier coefficients of a periodic function. In this section we'll consider the following calculation shortcuts for computing Fourier series:

1. Simplify computations for even or odd periodic functions. (Already covered in the previous topic.)

2. Use known Fourier series to compute the Fourier series for scaled and shifted functions.

3. Use known Fourier series to compute the Fourier series for the derivative or integrals of functions.

Even and odd functions were covered in the previous topic, so we won't go over them again here.

23.3.1 New series from old ones: shifting and scaling

First, if you scale and shift $f(t)$, then you scale and shift its Fourier series. To avoid burdening the statement with too much notation we state it for period 2π functions. You can extend this easily to any period.

Suppose $f(t)$ has Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$.

Theorem. (Scaling and shifting $f(t)$.) The scaled and shifted function $g(t) = cf(t) + d$ has Fourier series

$$g(t) \;=\; cf(t) + d \;=\; \frac{ca_0}{2} + d + \sum_{n=1}^{\infty} ca_n \cos(nt) + \sum_{n=1}^{\infty} cb_n \sin(nt).$$

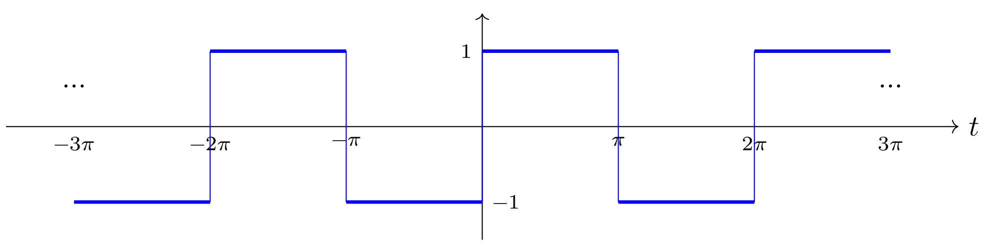
Theorem. (Scaling and shifting in time.) The function $g(t) = f(ct + d)$ has

$$g(t) \;=\; f(ct+d) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n(ct+d)) + \sum_{n=1}^{\infty} b_n \sin(n(ct+d)).$$

This is not quite in standard Fourier series form, but it is in a useable form. Also, if we really want a standard Fourier series, it is easy to expand out the trig functions to put it in standard form.

The rest of this subsection will be devoted to an extended example, illustrating these techniques using our standard period 2π square wave whose graph is shown just below.

Example 23.1. (Extended example.) The graph of $f(t)$ looks like this:



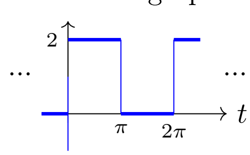
Graph of $f(t)$ = square wave

We know that the Fourier series for $f(t)$ is

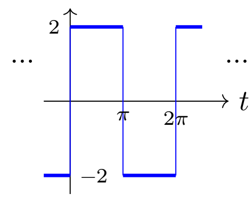
$$f(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}. \tag{1}$$

Now we will use this to find the Fourier series for scaled and shifted versions of $f(t)$. We'll define these new functions graphically, we could also write down formulas if we wanted.

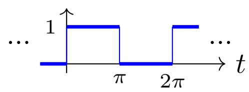
(a) $f_1(t) =$



$$\Rightarrow \; f_1(t) = 1 + f(t) = 1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}.$$

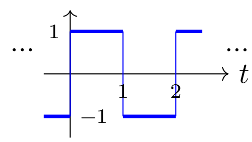
(b) $f_2(t) =$ 

$\Rightarrow f_2(t) = 2f(t) = \frac{8}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}$.

(c) $f_3(t) =$ 

$\Rightarrow f_3(t) = \frac{1}{2}(1 + f(t)) = \frac{1}{2} + \frac{2}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}$.

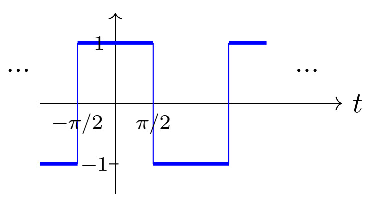
Next will look at what happens if we scale the time t.

(d) $f_4(t) =$ 

$\Rightarrow f_4(t) = f(\pi t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}$.

It's a little tricky to see that $f_4(t) = f(\pi t)$. I think about it two ways. First, the picture shows that we want $f_4(1) = f(\pi)$, which is given by $f_4(t) = f(\pi t)$. Second, $f_4(t)$ has period 2 so its Fourier series should have terms with frequencies $n\pi$.

Our last example involves shifting the time.

(e) $f_5(t) =$ 

$\Rightarrow f_5(t) = f(t + \pi/2) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n(t + \pi/2))}{n}$.

That is,

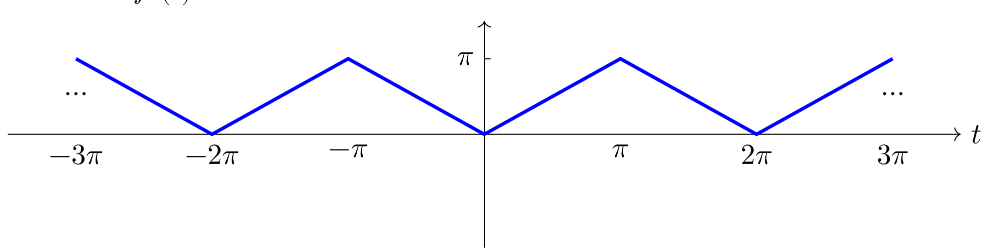
$$f_5(t) = \frac{4}{\pi}\left(\sin(t + \pi/2) + \frac{\sin(3t + 3\pi/2)}{3} + \ldots\right) = \frac{4}{\pi}\left(\cos t - \frac{\cos 3t}{3} + \ldots\right).$$

The last expression is in the form we defined for Fourier series. For most applications, the middle expression is perfectly useable and sometimes even preferable.

23.3.2 Differentiation and integration

If $f(t)$ is periodic, then the Fourier series for $f'(t)$ is just the term-by-term derivative of the Fourier series for $f(t)$. An example should make this clear.

Example 23.2. Let $f(t)$ be the period 2π triangle wave with $f(t) = |t|$ on $[-\pi.\pi]$. It's clear that $f'(t)$ is the square wave. Check that the derivative of the Fourier series of $f(t)$ is the Fourier series of $f'(t)$.

$f(t) =$ 

Graph of $f(t)$ = triangle wave

Solution: From the previous topic notes, we know the Fourier series for $f(t)$ is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \ldots\right)$$

Thus, $f'(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots\right)$. We know this is the Fourier series of our standard square wave as claimed.

Decay rate of Fourier series. Note that $f(t)$ has a corner and its coefficients decay like $1/n^2$, while $f'(t)$ has a jump and its coefficients decay like $1/n$. Note also, how differentiation changed the power of n in the decay rate.

Differentiation of discontinuous functions. Term-by-term differentiation of Fourier series works for discontinuous functions as long as we use the generalized derivative.

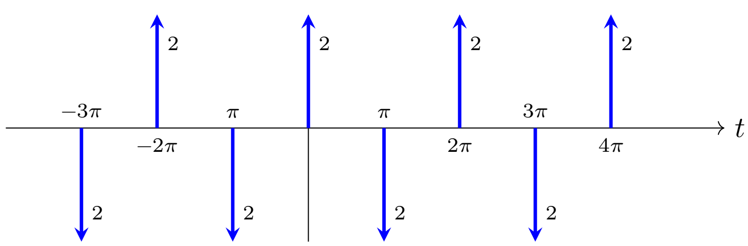
Example 23.3. Let $f(t)$ be our standard period 2π square wave. Find $f'(t)$ and the Fourier series of $f'(t)$. Graph $f'(t)$.

Solution: Because $f(t)$ has jumps (alternating between 2 and -2) we must take the generalized derivative:

$$f'(t) = \ldots - 2\delta(t+\pi) + 2\delta(t) - 2\delta(t-\pi) + 2\delta(t-2\pi) - \ldots$$

We know $f(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}$. So, taking the term-by-term derivative, $f'(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \cos(nt)$.

You can check this by computing the Fourier coefficients of $f'(t)$ directly using the integral formulas.



Graph of $f'(t)$ = impulse train

Example 23.4. Term-by-term integration. Suppose that

$$f(t) = 1 + \cos(t) + \frac{\cos(2t)}{2} + \frac{\cos(3t)}{3} + \frac{\cos(4t)}{4} + \ldots$$

What is $h(t) = \int_0^t f(u)\,du$?

Solution: We integrate the Fourier series term-by-term to get

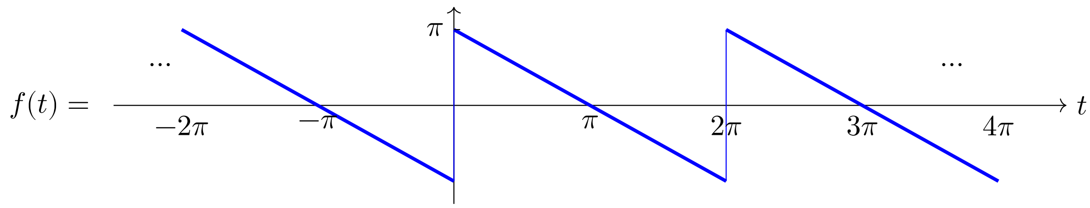
$$h(t) = \int_0^t f(u)\,du = C + t + \sin(t) + \frac{\sin(2t)}{2^2} + \frac{\sin(3t)}{3^2} + \ldots$$

Note: Just because $f(t)$ is periodic doesn't mean the integral of $f(t)$ will be periodic. In this case, the "t-term" shows that $h(t)$ is not periodic. So we can't officially say we have a Fourier series for $h(t)$. Nonetheless we have a nice series for $h(t)$ that can be used in many applications.

Here's one more example of integration. It's very cool, but we probably won't get to it in class.

Example 23.5. For your amusement. Consider the period 2π discontinuous sawtooth function

$$f(t) = \frac{\pi}{2} - \frac{t}{2} \quad \text{for } 0 < t < 2\pi.$$



Graph of $f(t)$ = discontinuous sawtooth

Since $f(t)$ is odd with period 2π, we know that the cosine coefficients $a_n = 0$. For the sine coefficients it is slightly easier to do the integral over a full period rather than double the integral over a half period:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - t}{2}\sin(nt)\,dt = \frac{1}{n}.$$

Thus, $f(t) = \sin(t) + \dfrac{\sin(2t)}{2} + \dfrac{\sin(3t)}{3} + \ldots.$

Now, let $h(t)$ be the integral of $f(t)$, specifically

$$\begin{aligned} \text{Let } h(t) &= \int_0^t f(u)\,du = \int_0^t \sin u + \frac{\sin 2u}{2} + \frac{\sin 3u}{3} + \ldots\, du \\ &= (1 - \cos(t)) + \frac{1 - \cos(2t)}{2^2} + \frac{1 - \cos(3t)}{3^2} + \ldots \\ &= \sum_1^\infty \frac{1}{n^2} - \sum_1^\infty \frac{\cos(nt)}{n^2}. \end{aligned}$$

The DC term is $\frac{a_0}{2} = \sum \frac{1}{n^2}$. This is an infinite sum, but we can compute its value directly using the integral formula for Fourier coefficients. On $[0, 2\pi]$, $h(t) = \displaystyle\int_0^t \frac{\pi}{2} - \frac{u}{2}\,du = \frac{\pi t}{2} - \frac{t^2}{4}$. Thus,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi t}{2} - \frac{t^2}{4}\,dt = \frac{\pi^2}{3}.$$

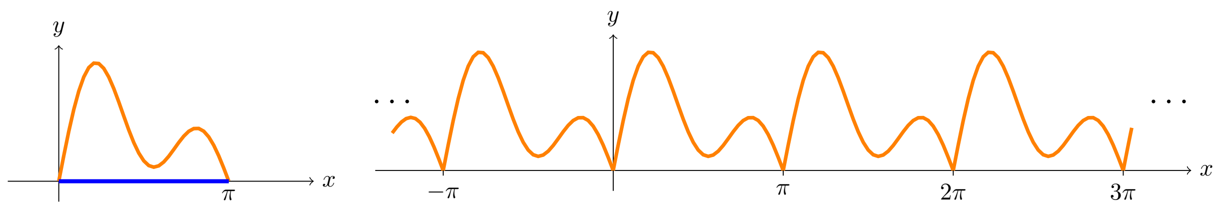
So, $\dfrac{a_0}{2} = \dfrac{\pi^2}{6} = \sum \dfrac{1}{n^2}$. We've summed an infinite series!

23.4 Sine and cosine series; even and odd extensions

23.4.1 Definition of sine and cosine series

In this section we will be concerned with functions $f(x)$ defined on an interval $[0, L]$. We start by stating the theorem on how to write functions as sine and cosine series. After that, we will use what we know about Fourier series to justify the theorem. We will need sine and cosine series when we study the heat and wave equations.

But first, an important semantic distinction: Fourier series are defined for periodic functions. A function defined only on an interval $[0, L]$ cannot be periodic, so it doesn't have a Fourier series. The figures below show a function defined on the interval $[0, \pi]$ and a period π function defined over the entire real line.



Left: function defined $[0, \pi]$, can't be periodic. Right: periodic function

Sine and cosine series. Without further ado, we state how to write a function as a cosine or sine series and how to compute the coefficients for the series. Note, the statements look very much like the ones for Fourier series.

Consider a function $f(x)$ defined on the interval $[0, L]$. $f(x)$ can be written as a cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad \text{where } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

$f(x)$ also has a sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Important.

1. Sine and cosine series are about functions defined on an interval.
2. The sine and cosine series have values for all x. At points in $(0, L)$ where $f(x)$ is continuous, the sine and cosine series equal $f(x)$. Since $f(x)$ is only defined on $[0, L]$, this is usually what we want.
3. Computing a_n and b_n only depends on the values of $f(x)$ in the interval $[0, L]$.
4. We will make use of sine and cosine series when we study the heat and wave equations.

23.4.2 Examples of sine and cosine series

Now, we'll give some example computations. We can do this by mechanically applying the formulas. We'll gain more insight into these series after we have seen the proof justifying the formulas for the coefficients.

Example 23.6. Find the Fourier cosine and sine series for the function $f(x) = \sin(x)$ defined on $[0, \pi]$.

Solution: Cosine series. $L = \pi$, Using the formula for a_n:

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin(x)\,dx = \left[-\frac{2}{\pi}\cos(x)\right|_0^\pi = \frac{4}{\pi}.$$

By applying the formula $\sin(a)\cos(b) = \dfrac{\sin(a+b)+\sin(a-b)}{2}$ we get:

$$a_n = \frac{2}{\pi}\int_0^\pi \sin(x)\,\cos(nx)\,dx = \frac{1}{\pi}\left[-\frac{\cos((1+n)x)}{1+n} - \frac{\cos((1-n)x)}{1-n}\right|_0^\pi = \begin{cases} 0 & \text{for odd } n > 0 \\ \frac{-4}{\pi(n^2-1)} & \text{for even } n > 0. \end{cases}$$

(You have to be careful with $n = 1$, but the formula is correct.)

Thus,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{3} + \frac{\cos(4x)}{15} + \frac{\cos(6x)}{35} + \ldots\right) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n>0,\ \text{even}} \frac{\cos(nx)}{n^2-1}.$$

Important. This is only valid where $f(x)$ is defined, i.e., on $[0,\pi]$.

Sine series. $f(x) = \sin(x)$ on $[0,\pi]$. This can be seen by comparing the abstract sine series $\sum_{n=1}^\infty b_n \sin(nx)$ with the given function $f(x) = \sin(x)$. Or we could compute the integrals for b_n similar to the way we computed a_n above.

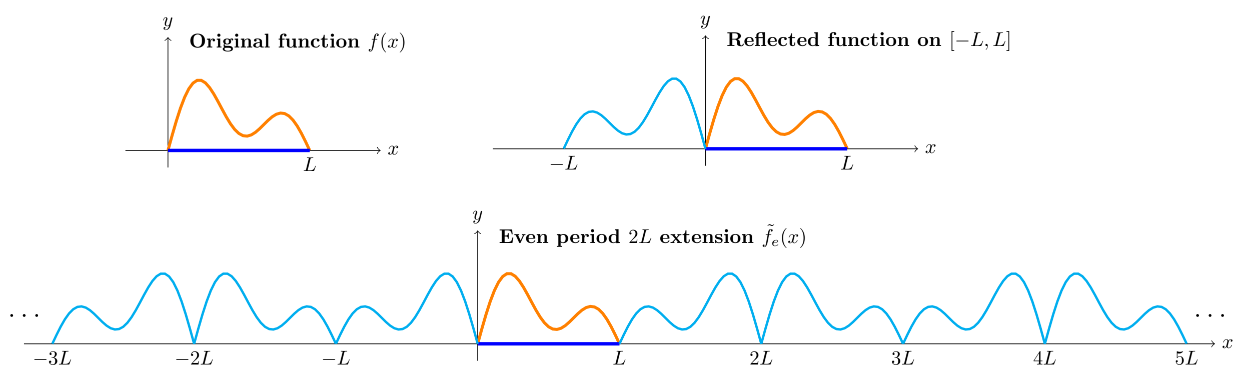
23.4.3 Even and odd periodic extensions

The proof of the formulas for the sine and cosine series coefficients turns out to be a straightforward application of Fourier series for periodic functions. The trick is to view the fact that $f(x)$ is only defined on $[0, L]$ as an opportunity instead of a limitation. To do this we need to define even and odd periodic extensions of $f(x)$.

Definition. If $f(x)$ is a function defined on the interval $[0, L]$ then the even period $2L$ extension of $f(x)$ is the period $2L$ function

$$\tilde{f}_e(x) = \begin{cases} f(-x) & \text{for } -L < x < 0 \\ f(x) & \text{for } 0 < x < L \end{cases}$$

To visualize this, we first reflect $f(x)$ in the y-axis to get a function defined over one period $[-L, L]$. We then extend this to be periodic over the entire real line.

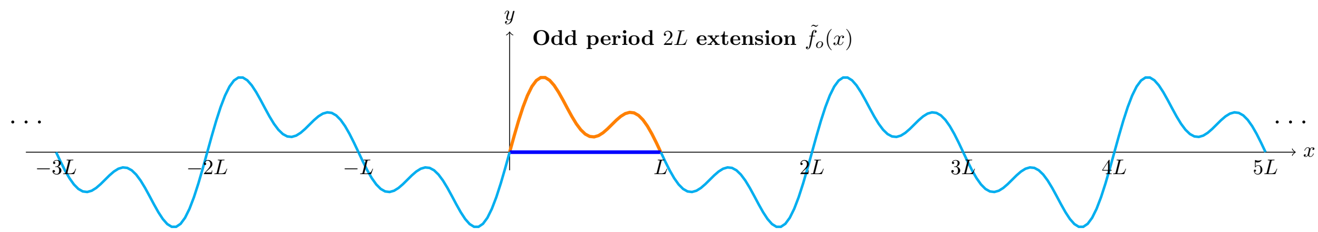


Making an even period $2L$ extension.

The odd period $2L$ extension of $f(x)$ is defined similarly, with

$$\tilde{f}_o(x) = \begin{cases} -f(-x) & \text{for } -L < x < 0 \\ f(x) & \text{for } 0 < x < L \end{cases}$$

To visualize this, we first reflect $f(x)$ through the origin to get a function defined over one period $[-L, L]$. We then extend this to be periodic over the entire real line.



The odd period $2L$ extension.

23.4.4 Proof of the formulas for the sine and cosine series

As we said, using the even and odd period $2L$ extensions this is a straightforward application of Fourier series for periodic functions. We will give the argument for the cosine series. The sine series is similar.

We have $f(x)$ defined on $[0, L]$ and the even period $2L$ extension $\tilde{f}_e(x)$. Since $\tilde{f}_e(x)$ is periodic, it has a Fourier series and since it is even this series has only cosine terms. That is,

$$\tilde{f}_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

Using the symmetry of even functions we know $a_n = \frac{2}{L}\int_0^L \tilde{f}_e(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$. But, on the interval of integration, we know $\tilde{f}_e(x) = f(x)$. Therefore,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

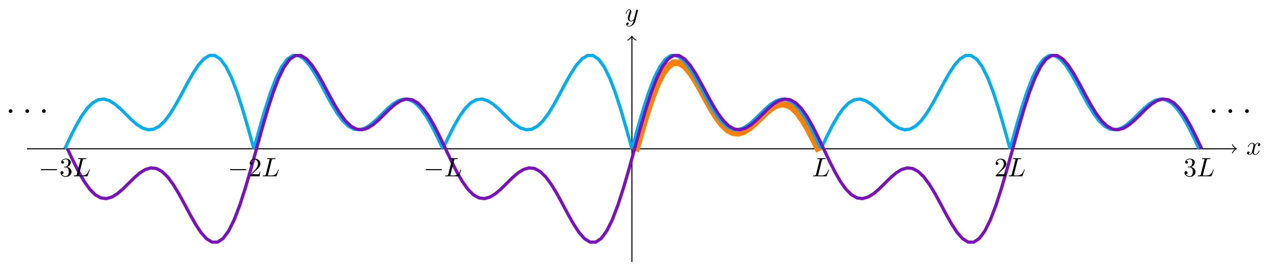
This is the formula we wanted to prove.

Sine series. You should try proving the formula for the sine series coefficients.

Once more to emphasize the grammar:
$f(x)$ is defined for x in $[0, L]$, while $\tilde{f}_e(x)$ and $\tilde{f}_o(x)$ are defined for all x.

The three functions agree on $[0, L]$, i.e., $f(x) = \tilde{f}_e(x) = \tilde{f}_o(x)$ for x in $[0, L]$. The cosine series for $f(x)$ is just the Fourier series for $\tilde{f}_e(x)$. The sine series for $f(x)$ is just the Fourier series for $\tilde{f}_o(x)$.

This is illustrated in the following figure:



$f(x)$ in orange, $\tilde{f}_e(x)$ in cyan, $\tilde{f}_o(x)$ in purple. All three are the same for $0 < x < L$.

We finish with an example that shows how to use known Fourier series to avoid computing integrals for sine and cosine series.

Example 23.7. Find the sine and cosine series for the function $f(x) = 1$ defined on the interval $[0, \pi]$.

Solution: Since the odd period 2π extension of $f(x)$ is our standard square wave, we have the sine series is the Fourier series of $\text{sq}(x)$:

$$f(x) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx)}{n}.$$

Since the even period 2π extension is the constant function $f(x) = 1$, we have the cosine series:

$$f(x) = 1.$$