ES.1803 Topic 24 Notes

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24 Linear ODEs with periodic input

24.1 Goals

1. Be able to solve a linear constant coefficient differential equation with periodic input by writing the input as a Fourier series.
   - Know to index the phase lags as $\phi(n)$ or ϕ_n in the superposition for the solution.
   - Be able to identify the term in the input that causes the biggest response.
   - Be able to recognize when one term in the Fourier series for the input produces a pure resonant term in the output.

24.2 Introduction

In this topic we combine Fourier series with the superposition principle to solve linear differential equations. This is really a small extension of what we did way back in the first unit where we had a finite number of terms being superpositioned. Now, with Fourier series, we have an infinite number of terms. Superposition works exactly the same way as before, but we'll have to work out how to present the solution in a nice form.

We'll do this by presenting a series of examples. You should pay attention to the format of the solutions.

24.3 Examples of constant coefficient DEs with periodic input

Example 24.1. Let $f(t)$ be the odd period 2π square wave with height 1. Find the periodic solution to the DE $\ddot{x} + 8x = f(t)$.

Solution: Using the (known) Fourier series for $f(t)$ the equation becomes

$$\ddot{x} + 8x = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}$$

In preparation for using the superposition principle we solve the DE separately for each term in the input, i.e., solve

$$\ddot{x}_n + 8x_n = \frac{\sin(nt)}{n}$$

The characteristic polynomial for this equation is $P(r) = r^2 + 8$. So,

$$P(in) = 8 - n^2; \quad |P(in)| = |8 - n^2|; \quad \text{Arg}(P(in)) = \boxed{\phi(n) = \begin{cases} 0 & \text{if } n \le 2 \\ \pi & \text{if } n \ge 3 \end{cases}}$$

The sinusoidal response formula gives us

$$x_{n,p}(t) = \frac{\sin(nt - \phi(n))}{n|P(in)|} = \frac{\sin(nt - \phi(n))}{n|8 - n^2|}.$$

Putting it together using superposition

$$x_p(t) = \frac{4}{\pi}\left(\frac{\sin(t)}{|8-1|} + \frac{\sin(3t-\pi)}{3|8-9|} + \frac{\sin(5t-\pi)}{5|8-25|} + ...\right) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt-\phi(n))}{n|8-n^2|}$$

We will often call this the steady periodic solution.

Important feature. Note that we were careful to label the phase lags as $\phi(n)$. This is because ϕ is be different for different terms. Instead of $\phi(n)$, we will sometimes use the notation ϕ_n.

Note: The solution given just above is correct, but we can make it a bit nicer looking by noting that $\sin(nt-\pi) = -\sin(nt)$. This gives us

$$x_p(t) = \frac{4}{\pi}\cdot\frac{\sin(t)}{7} - \frac{4}{\pi}\sum_{n \text{ odd},\, n\geq 3} \frac{\sin(nt)}{n|8-n^2|}.$$

Example 24.2. (Near resonance.) In the previous example, which term in the solution has the biggest amplitude?

Solution: The $n = 3$ term has the biggest amplitude (1/3). Note that the resonant frequency of the system is $\sqrt{8}$ and that 3 is the frequency in the Fourier series closest to this resonant frequency

Example 24.3. (Pure resonance.) Let $f(t)$ be the same square wave as in the previous examples. Find a particular solution to $\ddot{x} + 9x = f(t)$.

Solution: We solve the DE separately for each term in the input:

$$\ddot{x}_n + 9x_n = \frac{\sin(nt)}{n}$$

The characteristic polynomial is $P(r) = r^2+9$. The difference between this example and the previous one is that $P(3i) = 0$, so we will need the extended sinusoidal response formula.

$$P(in) = 9 - n^2; \quad |P(in)| = |9-n^2|; \quad \text{Arg}(P(in)) = \boxed{\phi_n = \begin{cases} 0 & \text{if } n \leq 2 \\ \pi & \text{if } n > 3 \\ \text{undefined} & \text{if } n = 3 \end{cases}}$$

For $n \neq 3$ the sinusoidal response formula gives us

$$x_{n,p}(t) = \frac{\sin(nt-\phi_n)}{n|P(in)|} = \frac{\sin(nt-\phi_n)}{n|9-n^2|}.$$

When $n = 3$, we have $P(3i) = 0$, so we'll need to use the extended SRF:

$$P'(r) = 2r \Rightarrow P'(3i) = 6i = 6e^{i\pi/2}.$$

So,

$$x_{3,p}(t) = \frac{t\sin(3t-\pi/2)}{3\cdot 6}$$

Putting it together, using superposition (and that $\sin(nt-\pi) = -\sin(nt)$), our solution is:

$$x_p(t) = \frac{4}{\pi}\left(\frac{\sin(t)}{8} - \frac{t\cos(3t)}{18} + \frac{\sin(5t-\pi)}{80} + \ldots\right) = \frac{4\sin(t)}{8\pi} - \frac{4t\cos(3t)}{18\pi} - \frac{4}{\pi}\sum_{n>3,\ n\text{ odd}} \frac{\sin(nt)}{n|9-n^2|}$$

Note: The input has angular frequency 1, but its Fourier series contains a frequency 3 component which causes pure resonance.

Example 24.4. Solve $\ddot{x} + 2\dot{x} + 9x = f$, where $f(t)$ is the triangle wave: $f(t) = |t|$ for $-\pi < t < \pi$.

Solution: We know from the Topic 22 notes that the Fourier series for $f(t)$ is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \ldots\right)$$

We'll use a slightly different pattern here and ignore the scale factors while we solve the DE for each term in the input. We'll bring the scale factors back when we use superposition. The DE for each piece is

$$\ddot{x}_n + 2\dot{x}_n + 9x_n = \cos(nt)$$

The characteristic polynomial is $P(r) = r^2 + 2r + 9$. So,

$$\begin{aligned}
P(in) &= 9 - n^2 + 2ni; \\
|P(in)| &= \sqrt{(9-n^2)^2 + 4n^2}; \\
\text{Arg}(P(in)) &= \boxed{\phi(n) = \tan^{-1}\left(\frac{2n}{9-n^2}\right) \text{ in Q1 or Q2.}}
\end{aligned}$$

Thus,

$$x_{n,p}(t) = \frac{\cos(nt-\phi(n))}{|P(in)|} = \frac{\cos(nt-\phi(n))}{\sqrt{(9-n^2)^2+4n^2}}.$$

There is also a constant term in the input, $\ddot{x}_0 + 2\dot{x}_0 + 9x_0 = \pi/2$. This is easy to solve:

$$x_{0,p}(t) = \frac{\pi}{18}.$$

Putting it together using superposition (and restoring the scale factors) our solution is:

$$\begin{aligned}
x_p &= x_{0,p} - \frac{4}{\pi}\left(x_{1,p} + \frac{x_{3,p}}{3^2} + \frac{x_{5,p}}{5^2} + \ldots\right) \\
&= \frac{\pi}{18} - \frac{4}{\pi}\left(\frac{\cos(t-\phi(1))}{\sqrt{68}} + \frac{\cos(3t-\phi(3))}{\sqrt{36}} + \frac{\cos(5t-\phi(5))}{\sqrt{356}} + \ldots\right) \\
&= \frac{\pi}{18} - \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\cos(nt-\phi(n))}{n^2\sqrt{(9-n^2)^2+4n^2}}
\end{aligned}$$

Note. The damping complicates the expressions for $P(in)$, but it also means that we don't need to worry about pure resonance.

We should do a first-order equation:

Example 24.5. Find the general solution to $\dot{x} + kx = f(t)$, where

$$f(t) = 1 + \frac{\cos(t)}{1} + \frac{\cos(2t)}{2} + \frac{\cos(3t)}{3} + ... = 1 + \sum_{n=1}^{\infty} \frac{\cos(nt)}{n}.$$

Solution: The problem asks for the general solution, so we start by giving the homogeneous solution: $x_h(t) = Ce^{-kt}$.

Finding x_p is similar to the examples above.

Characteristic polynomial: $P(r) = r + k$. So,

$$P(in) = k + in; \quad |P(in)| = \sqrt{k^2 + n^2}; \quad \text{Arg}(P(in)) = \boxed{\phi(n) = \tan^{-1}(n/k) \text{ in Q1}}.$$

Individual pieces: $\dot{x}_n + kx_n = \cos(nt)/n$. Using the SRF

$$x_{n,p}(t) = \frac{\cos(nt - \phi(n))}{n\sqrt{k^2 + n^2}}$$

Constant term: $\dot{x}_0 + kx_0 = 1 \Rightarrow x_{0,p} = 1/k$.

Now using superposition we find

$$\begin{aligned} x_p(t) &= x_{0,p} + x_{1,p} + x_{2,p} + x_{3,p} + ... \\ &= \frac{1}{k} + \frac{\cos(t - \phi(1))}{\sqrt{k^2 + 1}} + \frac{\cos(2t - \phi(2))}{2\sqrt{k^2 + 4}} + \frac{\cos(3t - \phi(3))}{3\sqrt{k^2 + 9}} + ... \\ &= \frac{1}{k} + \sum_{n=1}^{\infty} \frac{\cos(nt - \phi(n))}{n\sqrt{k^2 + n^2}} \end{aligned}$$

As always, the general solution is $x(t) = x_p(t) + x_h(t)$.