ES.1803 Topic 25 Notes

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25 PDEs; separation of variables

25.1 Goals

1. Be able to model the temperature of a heated bar using the heat equation plus boundary and initial conditions.
2. Be able to solve the equations modeling the heated bar using Fourier's method of separation of variables
3. Be able to model a vibrating string using the wave equation plus boundary and initial conditions.
4. Be able to solve the equations modeling the vibrating string using Fourier's method of separation of variables

25.2 Introduction

When a function depends on more than one variable, it has partial derivatives instead of ordinary derivatives. For 18.03, this means we will have to consider partial differential equations (PDE) involving such functions.

In this note we will focus on two main examples: the heat equation describing the temperature of heated metal rod and the wave equation describing the motion of a vibrating string We describe these below. In psets we will look at variations of these examples as well as extensions of our techniques to other equations.

Both examples lead to a linear partial differential equation which we will solve using the Fourier separation of variables method. Perhaps unsurprisingly, this will involve Fourier series, i.e., superposition of sines and cosines. Because there are multiple independent variables, the computations will be lengthier than we have seen before. However, the basic scheme will be the same. That is, to solve a homogeneous equation with initial conditions we:

1. Use the method of optimism to find modal solutions. In this case, there will be an infinite number of independent modal solutions.
2. The general solution is a linear combination of the modal solutions.
3. The values of the coefficients in the general solution are determined by the initial conditions. In this case, since we have an infinite number of terms in the linear combination, finding the coefficients will involve Fourier series.

For an inhomogeneous equation, the general solution is given by a particular solution plus the general homogeneous solution. We'll need some method, often the method of optimism, to find the particular solution.

The major new wrinkle will be the inclusion of what are called boundary conditions in our models. These will be explained in due course.

25.3 The heat equation

In this section we will look at the heat equation, which models the temperature over time in a heated bar.

Suppose we have a heated bar made of a uniform material. The temperature in the bar will vary with position along the bar as well as over time. To be specific, we assume we have a rod of length L which is thin enough that the temperature doesn't vary in the vertical direction. We will also make the assumption that the bar is insulated along its length so that no heat passes through the sides. (See the figure with the example in the next section.)

Given these assumptions, we can describe the temperature of the bar by a function of two variables $u(x, t)$ which gives the temperature at time t at position x.

The partial differential equation (PDE) modeling the temperature $u(x, t)$ is

$$\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u}{\partial x^2}(x,t). \tag{1}$$

Here, k is called the thermal conductivity of the material. It is a physical constant with dimension length^2/time.

Equation 1 is called the one-dimensional heat equation because it describes heat conduction in one dimension. The heat equation is ubiquitous in science and engineering. It models heat flow in a metal rod, diffusion of a contaminant in water, diffusion of information through a system and much more. It is a special case of an (in general nonlinear) equation called the diffusion equation.

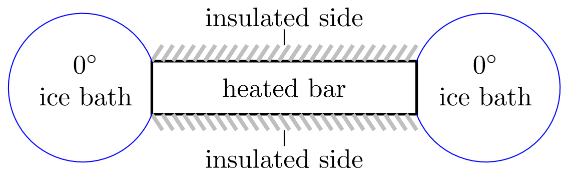
A nice derivation of the heat equation from physical principles is given in section 8.5 of the text by Edwards and Penney. Of course, you can also find many derivations on the internet.

25.3.1 Modeling a heated bar

We illustrate the modeling problem by going through one specific example. This is fairly wordy, but at the end of the example we will give a succinct summary of the model.

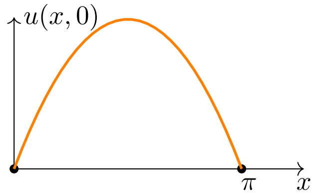
To be concrete, in this example, we'll use length in centimeters, temperature in degrees Celsius and time in seconds. We'll also let the thermal conductivity constant $k = 3$ cm/sec^2.

Suppose we have a thin heated bar of length $L = \pi$. We assume the top and bottom edges are insulated so that no heat passes through them. We also assume that the ends of the bars are in an ice bath maintained at 0°.



Over time, the temperature at various points along the bar will change. We let $u(x, t)$ be the temperature at the point x on the bar at time t.

Finally, we suppose that at time $t = 0$ the temperature over the bar is given by $u(x, 0) = x(\pi - x)$.



Initial temperature distribution (at $t = 0$).

The PDE modeling the temperature in a heated bar is given in Equation 1. With our value of k, this becomes

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 \leq x \leq \pi \text{ and } t > 0.$$

We want to finish the model for $u(x, t)$ by taking into account the ice baths and the initial temperature profile.

Since the ends of the bar are in ice baths held at 0°, we have the boundary conditions (BC)

$$u(0, t) = 0 \quad \text{and } u(\pi, t) = 0 \quad \text{for all } t.$$

The term boundary refers to the endpoints or *bounds* of the interval $[0, \pi]$. The boundary conditions (BC) give the values of $u(x, t)$ when x equals one of the bounds, i.e., $x = 0$ or $x = \pi$.

We are also given the temperature in the bar at time 0. This is called the initial condition (IC):

$$u(x, 0) = x(\pi - x).$$

We can summarize this as the heat equation with boundary and initial conditions:

- HE: $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$ for $0 \leq x \leq \pi, \quad t > 0$.
- BC: $u(0, t) = 0$ and $u(\pi, t) = 0$ for $t \geq 0$
- IC: $u(x, 0) = x(\pi - x)$ for $0 \leq x \leq \pi$.

25.3.2 A notational interlude

Using curvy d's to write partial derivatives is cumbersome and time consuming. Often we will use another standard notation for partial derivatives:

$$\frac{\partial u}{\partial x} = u_x, \quad \frac{\partial^2 u}{\partial x^2} = u_{xx} \quad \frac{\partial u}{\partial t} = u_t, \quad \frac{\partial^2 u}{\partial t^2} = u_{tt}.$$

With this notation our model becomes

- HE: $u_t = 3u_{xx}$ for $0 \leq x \leq \pi, \quad t > 0$.
- BC: $u(0, t) = 0$ and $u(\pi, t) = 0$ for $t > 0$.
- IC: $u(x, 0) = x(\pi - x)$ for $0 \leq x \leq \pi$.

25.3.3 A strategy for solving the heat equation with boundary and initial conditions

To solve the system described above means finding a function $u(x, t)$ that satisfies all three of the criteria: HE, BC, IC. Our strategy will start by ignoring the initial condition.

1. First, we'll use the method of optimism to find simple (modal) solutions that satisfy both the partial differential equation (HE) and the boundary conditions (BC).
2. The general solution satisfying the HE and the BC will be the superposition of all the modal solutions.
3. Finally, the initial condition (IC) will let us determine the values of the coefficients in the general solution.

This outline should look familiar: it's exactly the same as the outline we used for solving linear homogeneous differential equations $P(D)x = 0$. The details of the computation will of course be different.

Before going into these details we need to check linearity and homogeneity.

25.3.4 The heat equation is linear and homogeneous

In this part we will give a quick argument showing that the heat equation is linear and homogeneous. Here's one way of explaining what we mean:

First we rewrite the heat equation to bring out the homogeneity:

$$u_t - 3u_{xx} = 0.$$

Now we define the heat operator H by $Hu = u_{tt} - 3u_{xx}$. Remember that the notation Hu should be read as 'H applied to u'. With this notation the heat equation is simply $Hu = 0$.

As usual, once we realize the need, showing that the operator H is linear is some simple algebra. That is, we must show that

$$H(c_1 u_1 + c_2 u_2) = c_1 H u_1 + c_2 H u_2$$

for any constants c_1, c_2. Since this is just the statement that taking partial derivatives is linear we leave it to you to verify.

This is important. Linearity is important in 18.03. You should make extra certain that you understand what is being said in this section. If it's not clear, make sure to keep asking questions until it is!

Being linear and homogeneous, linear combinations of solutions to the heat equation are also solutions, i.e., if $Hu_1 = 0$ and $Hu_2 = 0$, then $H(c_1 u_1 + c_2 u_2) = 0$.

25.3.5 The boundary conditions are linear and homogeneous

By linear and homogeneous boundary conditions, we mean that if two functions $u_1(x, t)$ and $u_2(x, t)$ satisfy the boundary conditions then so does any linear combination of u_1 and

u_2. This should be clear for the boundary conditions from our example: $u(0,t) = 0$ and $u(\pi, t) = 0$.

This is important redux. Linearity is important in 18.03. You should make extra certain that you understand what is being said in this section. If it's not clear make sure to keep asking questions until it is!

Note. If the boundary conditions were not 0, then they would be linear but not homogeneous. You should be able to formulate the superposition principle that they satisfy in this case.

25.4 Solving the head equation with boundary and initial conditions

We are almost ready to learn the Fourier separation of variables method. Now might be a good time to review the strategy described in the Section 25.3.3.

25.4.1 Preliminary notions

Once we get going, we will need the following notions.

Notion 1. The ordinary differential equation $X''(x) + \lambda X(x) = 0$ has 3 cases:

Case (i) If $\lambda > 0$, the solution is $X(x) = a\cos(\sqrt{\lambda}\, x) + b\sin(\sqrt{\lambda}\, x)$

Case (ii) If $\lambda = 0$, the solution is $X(x) = a + bx$

Case (iii) If $\lambda < 0$, the solution is $X(x) = ae^{\sqrt{-\lambda}\, x} + be^{-\sqrt{-\lambda}\, x}$

Notion 2. If x and t are independent variables and $f(x)$ and $g(t)$ are functions with $f(x) = g(t)$ for all x and t, then both $f(x)$ and $g(t)$ are constant functions equal to the same constant.

To wrap your mind around what is being said, you should focus on the fact that x and t are independent. This means that we can fix $x = 2$ and let t vary. So, under the assumption that $f(x) = g(t)$ for all x and t, we have

$$f(2) = g(t) \text{ for all } t.$$

Since $f(2)$ is a constant, this means that $g(t)$ is a constant function. The argument that $f(x)$ is a constant function is identical. Clearly, they both equal the same constant.

Notion 3. The function $u(x,t) \equiv 0$ satisfies both HE and BC. We call this the trivial solution. While it is a fine upstanding solution, it won't be much help in our search for modal solutions that can be used in linear combinations.

25.4.2 Fourier's method of separation of variables

We now return to our example: Our first version of the solution will be rather long-winded because we will need to explain each step in the method. Later, we will be able to give more streamlined solutions.

Example 25.1. Solve the following partial differential equation (PDE) with boundary and initial conditions (BC & IC). That is, find a function $u(x,t)$ that satisfies the following.

- HE: $u_t = 3u_{xx}$, where $0 \le x \le \pi$ and $t > 0$
- BC: $u(0,t) = 0$ and $u(\pi,t) = 0$, where $t > 0$
- IC: $u(x,0) = x(\pi - x)$, where $0 \le x \le \pi$.

Solution: Step 1 Separated solutions. Our first trick is to use the method of optimism to look for a solution of the form

$$u(x,t) = X(x)T(t).$$

This is called a separated solution because it is a function of x times a function of t. There is no reason to expect that *all* solutions are separated, but that doesn't mean we won't find some useful solutions this way.

Having guessed a trial solution, we substitute it into the partial differential equation HE. This gives:

$$X(x)T'(t) = 3X''(x)T(t).$$

Now we separate the equation so the x's are on one side and the t's are on the other

$$\frac{T'(t)}{3T(t)} = \frac{X''(x)}{X(x)}.$$

Note, the convention is to keep the coefficient 3 with the T. Please do this, it will make your life easier. Now, preliminary Notion 2, comes into play: the left side is a function of t and the right side is a function of x, so both must be constant functions equal to the same constant!

We can call this constant anything we want. Because we know it will help with the algebra that is coming, we call it $-\lambda$. This too is just a convention, but you should do it so λ will mean the same thing for everyone in ES.1803. We have

$$\frac{X''(x)}{X(x)} = -\lambda; \qquad \frac{T'(t)}{3T(t)} = -\lambda.$$

A tiny bit of algebra gives the two ordinary differential equations

$$X''(x) + \lambda X(x) = 0; \qquad T'(t) + 3\lambda T(t) = 0.$$

Now we appeal to our preliminary Notion 1 to look at the 3 cases for λ. In all three cases $u(x,t) = X(x)T(t)$.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = e^{-3\lambda t}$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c$.

Case (iii) $\lambda < 0$: $X(x) = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$, $T(t) = ce^{-3\lambda t}$.

Case (iii) is ugly. Notice that $-\lambda$ is positive so the square roots are real numbers and so these are actually real-valued solutions. Fortunately, we will soon see that we can ignore it since it only gives the trivial solution satisfying the partial differential equation HE and the boundary conditions BC.

The method of optimism was wildly successful. We have lots of solutions to HE. We can get a separated solution to HE by picking any value of λ and then any values for a, b, c.

Step 2 Boundary conditions (BC). The model also has boundary conditions. So we need to see which of the separated solutions to the partial differential equation HE also satisfy the boundary conditions BC. Such solutions are called modal solutions.

For a separated solution $u(x,t) = X(x)T(t)$, the boundary conditions are

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(\pi,t) = X(\pi)T(t) = 0.$$

Being extra careful: this means that either $X(0) = X(\pi) = 0$ or $T(t) = 0$. The case $T(t) = 0$ gives the trivial solution $u(x,t) = X(x)T(t) = 0$. Since it is trivial, we ignore this case. So (nontrivial) separated solutions satisfying both HE and BC must have

$$X(0) = 0 \quad \text{and} \quad X(\pi) = 0. \tag{2}$$

Now we'll look at each case in turn.

Case (i). $\lambda > 0$: $X(x) = (a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x))$. The boundary conditions give

$$X(0) = a = 0 \quad \text{and} \quad X(\pi) = a\cos(\sqrt{\lambda}\pi) + b\sin(\sqrt{\lambda}\pi) = 0.$$

Solving, we see that $a = 0$ and either $b = 0$ or $\sin(\sqrt{\lambda}\pi) = 0$. The choice $b = 0$ gives us the trivial solution, so we ignore it. The other choice, $\sin(\sqrt{\lambda}\pi) = 0$ gives $\sqrt{\lambda}\pi = n\pi$ for some integer n. So, for each $\sqrt{\lambda} = n$ $(\lambda = n^2)$, we have the following separated solutions that satisfy both HE and BC:

$$u_n(x,t) = X_n(x)T_n(t), \text{ where } X_n(x) = b_n\sin(nx), \text{ and } T_n(t) = c_n e^{-3n^2t}.$$

Note, we name the solutions and coefficients with the subscript n so we can tell them apart.

A simplification: in the product we can combine b_n into c_n one constant so,

$$\boxed{u_n(x,t) = b_n\sin(nx)\,e^{-3n^2t} \quad \text{for } n = 1, 2, 3, \ldots}$$

are the separated solutions for case (i) which satisfy both HE and BC.

In this case, the boundary condition weeded out most values of $\lambda > 0$.

Case (ii). $\lambda = 0$: $X(x) = a + bx$. The boundary conditions are

$$X(0) = a = 0 \qquad X(\pi) = a + b\pi = 0.$$

It is easy to see that the only solutions to these equations are $a = 0$, $b = 0$. That is, this case only produces trivial solutions and we can ignore it.

Note well: With other boundary conditions this case may produce nontrivial solutions. So we always have to check.

Case (iii). $\lambda < 0$: $X(x) = ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x}$. The boundary conditions are

$$X(0) = a + b = 0 \qquad X(\pi) = ae^{\sqrt{-\lambda}\,\pi} + be^{-\sqrt{-\lambda}\,\pi} = 0.$$

In matrix form the equation is

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\,\pi} & e^{-\sqrt{-\lambda}\,\pi} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix is $e^{-\sqrt{-\lambda}\,\pi} - e^{\sqrt{-\lambda}\,\pi}$. Since $\lambda \neq 0$ the determinant is not 0. Therefore, we only have the trivial solution $a = 0$, $b = 0$. That is, this case only yields the trivial solution to HE and BC. So we ignore it!

It turns out, this case will never give nontrivial solutions. So this is the first and last time we will do the algebra for this case. **In the future we will just say that Case (iii) only has the trivial solution and ignore it.**

Note. All the separated solutions satisfying both HE and BC are called normal modes or modal solutions for this system.

We have now found all the modal solutions.

Step 3 Superposition. Because both the PDE (HE) and the boundary conditions (BC) are linear and homogeneous, the general solution satisfying them both is given by superposition of all the modal solutions:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-3n^2 t}}.$$

Step 4 Use the IC to find the coefficients. We are now ready to use the initial conditions (IC) to determine the values of the coefficients b_n in our general solution.

IC: $u(x,0) = \sum b_n \sin(nx) = x(\pi - x)$. Therefore, b_n are the Fourier sine coefficients of $x(\pi - x)$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\, dx = \begin{cases} 8/(\pi n)^3 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

(The full computation of this integral is shown in the section at the end of these notes.) Thus, our solution is

$$\boxed{u(x,t) = \sum b_n \sin(nx)\, e^{-3n^2 t} = \sum_{n \text{ odd}} \frac{8}{(n\pi)^3} \sin(nx)\, e^{-3n^2 t}.}$$

25.5 Summary of Fourier's method

Once again we summarize Fourier's method for homogeneous PDEs with homogeneous boundary conditions.

1. Find separated solutions to the PDE: one parametrized family for each λ.

2. The boundary conditions (BC) restrict the λ to an indexed set of values. They also restrict the possible values of the parameters in each family.

3. Superposition gives the general solution satisfying both the PDE and BC.

4. Use the initial conditions to determine the values of the coefficients in the general solution..

25.6 Model solution

Because the first time through took several pages, we redo the solution to the previous example in model form. But this comes with a **WARNING**: do not just memorize this routine. You should remember the reasons for each of the steps. Different problems will use variations on these themes and you have to be prepared to use the reasoning, but not the exact details, from this example.

Example 25.2. (Model solution.) Solve for $u(x,t)$ on $0 \le x \le \pi$ and $t > 0$ satisfying

- **HE:** $u_t = 3u_{xx}$.
- **BC:** $u(0,t) = 0$ and $u(\pi,t) = 0$.
- **IC:** $u(x,0) = x(\pi - x)$.

Solution: Step 1. Look for separated solutions: $u(x,t) = X(x)T(t)$ to the PDE.

Substitution into HE: $XT' = 3X''T$.

Algebra: $X''(x)/X(x) = T'(t)/(3T(t)) = \text{ constant } = -\lambda$.

More algebra: $X'' + \lambda X = 0, \quad T' + 3\lambda T = 0$. There are three cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x), \quad T(t) = ce^{-3\lambda t}$.

Case (ii) $\lambda = 0$: $X(x) = a + bx, \quad T(t) = c$.

Case (iii) $\lambda < 0$. Always ignore, since this case only gives the trivial solution satisfying the PDE and boundary conditions.

Step 2. Modal solutions. Find the separated solutions which also satisfy the BC.

For separated solutions, the BC are $\quad X(0) = 0, \quad X(\pi) = 0$.

Case (i) The BC are

$$X(0) = a = 0 \quad \text{and} \quad X(\pi) = a\cos(\sqrt{\lambda}\pi) + b\sin(\sqrt{\lambda}\pi) = 0.$$

Since $a = 0$, the second condition is $b\sin(\sqrt{\lambda}\pi) = 0$. For nontrivial solutions, we need $\sqrt{(}\sqrt{\lambda}\pi) = 0$, i.e., $\sqrt{\lambda}\pi = n\pi$ for n an integer.

We have found modal solutions

$$u_n(x,t) = b_n \sin(nx)\, e^{-3n^2 t} \qquad \text{for } n = 1, 2, 3, \ldots$$

Case (ii) The BC are $X(0) = a = 0$, $X(\pi) = a + b\pi = 0$.

This has only the trivial solution $a = 0$, $b = 0$.

Case (iii) Ignored – only has the trivial solution.

Step 3. Both HE and BC are homogeneous, so, by superposition, the general solution satisfying both is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-3n^2 t}}.$$

Step 4. Use the initial conditions to find the values of the coefficients.

IC: $u(x,0) = \sum_{n=1} b_n \sin(nx) = x(\pi - x)$. This is the Fourier sine series for $x(\pi - x)$. Now (see the computation section below) the coefficients are

$$\boxed{b_n = \begin{cases} 8/(\pi n)^3 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}}$$

So,

$$\boxed{u(x,t) = \sum_{n \text{ odd}} \frac{8}{\pi n^3} \sin(nx)\, e^{-3n^2 t}}.$$

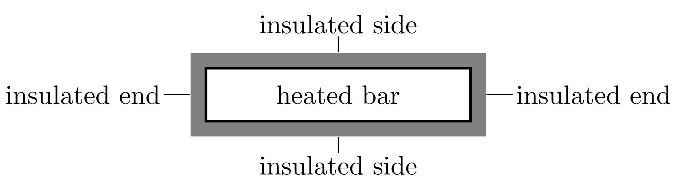
25.7 Another example with different boundary conditions

Here's an example with different boundary conditions. In this example, we will see that the case $\lambda = 0$ has nontrivial solutions.

Example 25.3. Suppose we have a heated rod of length L as described above. Assume that the ends of the bar are also insulated, so that no heat leaves the bar. Also assume that the initial temperature of the bar is given by $u(x,0) = x(L-x)\,^\circ C$.

Write down a PDE with boundary and initial conditions that models the temperature in the bar. Then use Fourier's separation of variables method to solve the system.

Solution: The physical setup is illustrated in the figure below.



Heated rod insulated on all sides

First we set up the model: The PDE is just the heat equation given in (1):

(HE) $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{for } t > 0 \text{ and } 0 \le x \le L.$$

We are not given enough information to determine k, so we leave it as an unspecified parameter.

Because the ends of the rod are insulated, the temperature gradient at the ends is 0. This translates to the boundary conditions:

(BC) $$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0 \quad \text{for } t > 0.$$

Note that these are homogeneous boundary conditions.

We are given the initial condition (i.e., the temperature at time 0) directly:

(IC) $$u(x,0) = x(L-x) \quad \text{for } 0 \le x \le L.$$

Next we solve using the method of separation of variables.

Step 1. Look for separated solution: $u(x,t) = X(x)T(t)$ to the PDE.

Substitution into HE: $XT' = kX''T$.

Algebra: $X''(x)/X(x) = T'(t)/(kT(t)) =$ constant $= -\lambda$.

More algebra: $X'' + \lambda X = 0, \quad T' + k\lambda T = 0$. There are three cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = ce^{-k\lambda t}$.

Case (ii) $\lambda = 0$: $X(x) = a + bx, \quad T(t) = c$.

Case (iii) $\lambda < 0$. Always ignore, since this case only gives the trivial solution satisfying the PDE and boundary conditions.

Step 2. (Modal solutions) Find the separated solutions in Step 1 which also satisfy the boundary conditions.

For separated solutions, the BC are $X'(0) = 0, \quad X'(L) = 0$.

Case (i) $X'(0) = \sqrt{\lambda}b = 0$ and $X'(L) = -\sqrt{\lambda}a\sin(\sqrt{\lambda}L) + \sqrt{\lambda}b\cos(\sqrt{\lambda}L)$.

This has nontrivial solutions when $b = 0$ and $\sqrt{\lambda}L = n\pi$ for n an integer. That is, when $\sqrt{\lambda} = n\pi/L$. For this case, the modal solutions are

$$u_n(x,t) = a_n \cos\left(\frac{n\pi}{L}x\right) e^{-k(n\pi/L)^2 t} \qquad \text{for } n = 1, 2, 3, \ldots$$

Case (ii) $X'(0) = b = 0$, $X'(L) = b = 0$. This has nontrivial solutions $X(x) = a$. So, for this case, the modal solutions are $X(x)T(t) = ac$. We write this as

$$u_0(x,t) = \frac{a_0}{2}.$$

Step 3. Both HE and BC are homogeneous, so by superposition the general solution satisfying both is

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L}x\right) e^{-k(n\pi/L)^2 t}.}$$

(We gave the DC term as $a_0/2$ so we don't forget the factor of $1/2$ when we do the computation below.)

Step 4. Use the initial conditions to find the coefficients.

$u(x,0) = \dfrac{b_0}{2} + \sum_{n=1} b_n \cos\left(\dfrac{n\pi}{L}x\right) = x(L-x)$. This is the Fourier cosine series for $x(L-x)$.

Now (see the computation section below) the coefficients are

$$\boxed{b_0 = L^2/3, \quad b_n = \begin{cases} -4L^2/(n\pi)^2) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}}$$

We can leave our answer as a set of boxes or put them together in one box.

$$\boxed{u(x,t) = \frac{L^2}{6} - \sum_{n \text{ even}} \frac{4L^2}{(n\pi)^2} \cos\left(\frac{n\pi}{L}x\right) e^{-k(n\pi/L)^2 t}.} \qquad (3)$$

25.7.1 Interpreting the solution

Writing out the terms in the solution Equation 3 above we have

$$u(x,t) = \frac{L^2}{6} - \frac{4L^2}{(2\pi)^2}\cos\left(\frac{2\pi}{L}x\right)e^{-k(2\pi/L)^2 t} - \frac{4L^2}{(4\pi)^2}\cos\left(\frac{4\pi}{L}x\right)e^{-k(4\pi/L)^2 t}$$
$$- \frac{4L^2}{(6\pi)^2}\cos\left(\frac{6\pi}{L}x\right)e^{-k(6\pi/L)^2 t}\ldots$$

The first thing to note is that all the terms after the constant have decaying exponentials in time. This means that, in the long run, the bar will come to an equilibrium temperature of $L^2/6$. It makes intuitive sense that the temperature in the bar will even out over time. Looking at the expression for the constant (DC) term

$$\frac{c_0}{2} = \frac{1}{L}\int_0^L x(L-x)\,dx$$

we see that it is the average value of the initial temperature distribution. This too makes intuitive sense.

The second thing we want to note is that, after a very short time, the solution is well approximated by the DC term and the first non-zero harmonic

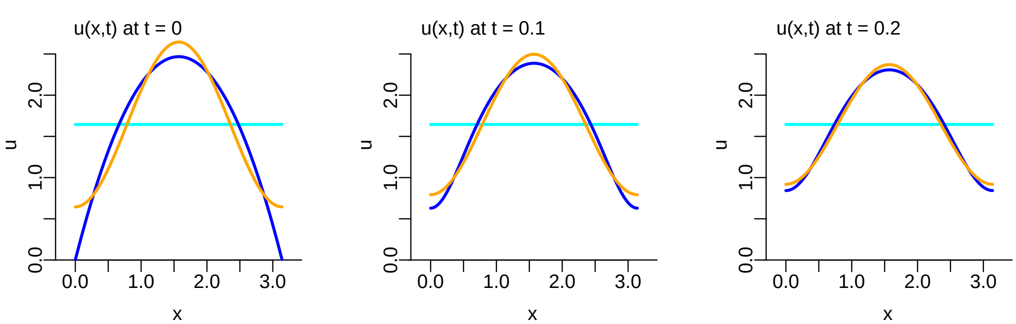
$$u(x,t) \approx \frac{L^2}{6} - \frac{4L^2}{(2\pi)^2}\cos\left(\frac{2\pi}{L}x\right)e^{-k(2\pi/L)^2 t}$$

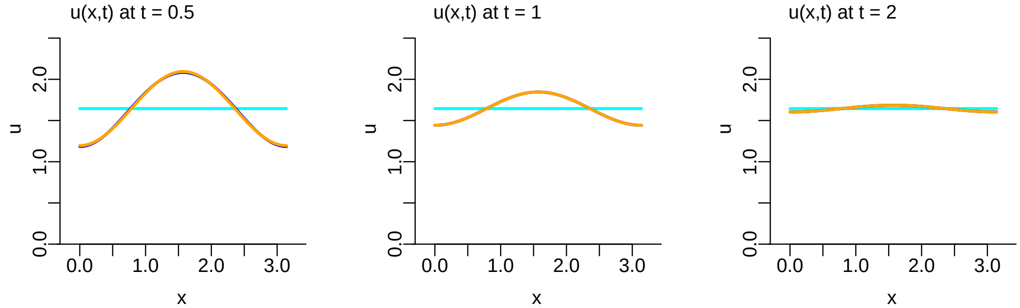
To see this look at the exponents in the time exponentials:

$$e^{-\frac{k4\pi^2}{L^2}t}, \quad e^{-\frac{k16\pi^2}{L^2}t}, \quad e^{-\frac{k36\pi^2}{L^2}t}, \ldots$$

The later exponents are so much more negative than the first one that the later exponentials rapidly become negligible compared to the first.

Here are a sequence of plots showing the exact solution, the long term equilibrium and the approximation by the DC term plus the first non-zero harmonic. (They were made with $L = \pi$ and $k = 0.4$.) Notice how well the approximation matches the exact solution after a short time. Also notice how the solution goes to the equilibrium over time.





Blue = exact sol., cyan = equilibrium, orange = $\frac{L^2}{6} - \frac{4L^2}{(2\pi)^2}\cos\left(\frac{2\pi}{L}x\right) e^{-k(2\pi/L)^2 t}$

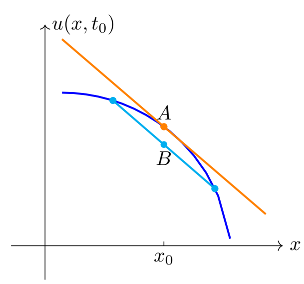
As usual, there is an applet giving a dynamic illustration of the heat equation: https://mathlets.org/mathlets/heat-equation/.

25.7.2 A mathematical explanation of the heat equation

Suppose the temperature along the bar is given by $u(x,t)$. That is, this is the temperature at point x at time t.

Let's fix a time t_0 and a position x_0. It's reasonable to assume that if the average temperature of nearby points is lower than that at x_0, then the point at x_0 will be losing heat, i.e., the rate the temperature changes will be negative. Likewise, if the average temperature of nearby points is greater than that at x_0 then the rate the temperature changes will be positive.

The graph below shows the temperature distribution $u(x,t_0)$ at a single time t_0. The temperature at x_0 is marked by the point A on the curve. The average temperature of the two points (equally spaced around x_0) is shown as the point B on the secant line between the two points. Since the curve is concave down, this average B is below A, i.e., the average temperature is lower than the temperature at x_0. So the rate, $\frac{\partial u}{\partial t}(x_0,t_0)$, the temperature is changing is negative.



The concavity of the curve at x_0 is measured by $\frac{\partial^2 u}{\partial x^2}(x_0,t_0)$. Since the curve is concave down, this is negative, i.e., the same sign as the rate the temperature is changing. Thus, at least to a first approximation, the rate the temperature is changing is given by the heat equation

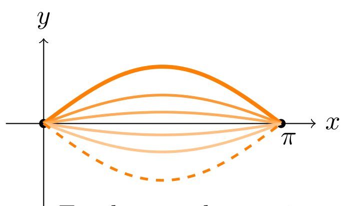
$$\frac{\partial u}{\partial t}(x_0,t_0) = k\frac{\partial^2 u}{\partial x^2}(x_0,t_0).$$

25.8 Physics of a vibrating string: the wave equation

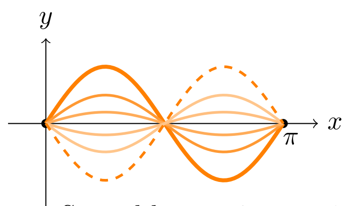
Suppose we have a string or wire of length L tied tightly between two posts, If we start it vibrating it will give off a sound. That is, it will create a pressure wave that will strike our ears and we will hear a sound.

If we are careful, we can make the starting shape a perfect sine curve and then, when we let the string go, it will vibrate while always maintaining its sine curve shape with only the amplitude changing in time. (This is not at all obvious, at least to me, but it will come out in our analysis of the wave equation.) This perfect sine curve shaped vibration is called a normal or pure mode. In this mode the string will emit a pure tone. The twanginess of most vibrating strings tells you that they do not spontaneously vibrate in a normal mode.

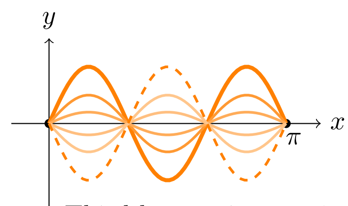
For $L = \pi$ the first 3 normal modes have starting shape $y = b_1 \sin(x)$, $y = b_2 \sin(2x)$ and $y = b_3 \sin(3x)$. These are called the first or fundamental harmonic, second harmonic and third harmonic respectively. We illustrate this in the pictures below.



Fundamental: $n = 1$



Second harmonic: $n = 2$

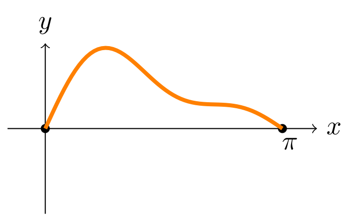


Third harmonic: $n = 3$

An even better way to visualize normal modes is to go to the applet `https://mathlets.org/mathlets/wave-equation/`. Refresh the page so the initial condition is set to its default. Set $n = 3$, and leave only the *harmonics* checkbox checked. Then start the animation by clicking the >> button. You will see each of the modes vibrating. Then check the *Fourier sum* checkbox and you will see the superposition of the 3 harmonics.

The twanginess of a vibrating string comes because the vibration is really a superposition or mixture of the many normal modes. The figure shows the starting position for a mixture of the first 3 modes (fundamental, first and second harmonics) with amplitudes 1, 0.5 and 0.3 respectively. That is

$$y = \sin(x) + 0.5\sin(2x) + 0.3\sin(3x)$$



Mixture: $y = \sin(x) + 0.5\sin(2x) + 0.3\sin(3x)$

25.8.1 Physical model of a vibrating string

This is well explained in just 2 paragraphs in §8.6 of the textbook by Edwards and Penney. They then derive the partial differential equation modeling the vibrating string. We give a quick summary of the terminology and model here. You should read the text or look on the internet to see the derivation.

The physical assumption is that each point on the string only moves up and down in the y-direction, i.e., there is no side-to-side movement. This and their other assumptions are reasonable for strings that are much longer than the amplitude of their vibration.

For a point x on the string we let

$$y(x,t) = \text{ displacement of the point } x \text{ at time } t.$$

Assuming small displacements, this is well modeled by the following partial differential equation, called the wave equation

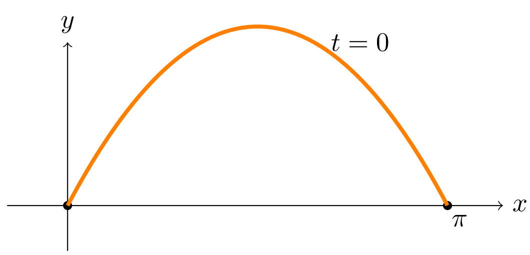
$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{4}$$

Here a is a constant which depends on the physical characteristics of the string as well as its tension and length, it is the called the speed of the wave. You should check that it does indeed have units of speed.

25.9 The wave equation with boundary and initial conditions.

25.9.1 Modeling a vibrating string

We illustrate the modeling problem by going through one specific example. Suppose we have a string of length $L = \pi$ meters which is clamped at both ends. As the string vibrates, let $y(x,t)$ be the displacement in meters of the point x on the string at time t in seconds. Suppose at time $t = 0$ the string is stationary and has shape $y(x,0) = x(\pi - x)$.



Initial shape of the string (at $t = 0$).

We want to find a model for $y(x,t)$. The model will consist of a partial differential equation (PDE) and some extra conditions. For this example, assume the wave speed is 3 m/sec.

Above we asserted that the PDE modeling a vibrating string is given in Equation 4. With our units this becomes

$$\frac{\partial^2 y}{\partial t^2} = 9 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 \le x \le \pi \text{ and } t > 0.$$

Since the ends are clamped they cannot move. That is, the points on the string at $x = 0$ and $x = \pi$ are fixed, i.e., we have the boundary conditions (BC)

$$y(0,t) = 0 \qquad \text{and} \qquad y(\pi,t) = 0 \qquad \text{for all } t.$$

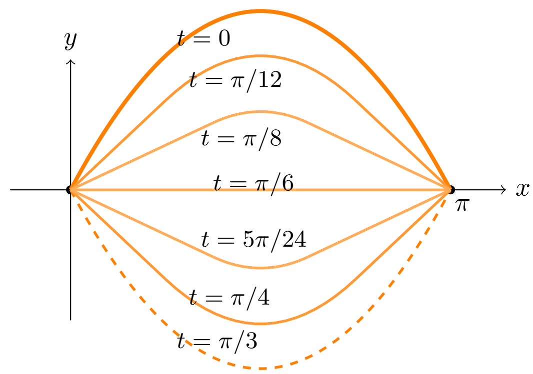
We are also given the shape (displacement) and velocity of the string at time 0. These are the initial conditions (IC). The initial shape was given as $x(\pi - x)$. The initial velocity is 0, because the string is momentarily stationary at $t = 0$. Since shape at time 0 is $y(x, 0)$ and the velocity is $\frac{\partial y}{\partial t}(x, 0)$ we have the initial conditions

$$y(x,0) = x(\pi - x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = 0.$$

We can summarize this as the wave equation with boundary and initial conditions:

- WE: $\dfrac{\partial^2 y}{\partial t^2} = 9\dfrac{\partial^2 y}{\partial x^2}$ for $0 \le x \le \pi, \quad t > 0$.
- BC: $y(0,t) = 0$ and $y(\pi, t) = 0$ for $t \ge 0$
- IC: $y(x,0) = x(\pi - x)$ and $\frac{\partial y}{\partial t}(x,0) = 0$ for $0 \le x \le \pi$.

The figure below shows that shape of the string at various points in time. Note that the boundary points don't move because of the clamped end boundary conditions.



25.9.2 Solving the wave equation with boundary and initial conditions

As with our heat equation examples, we will use Fourier's method of separation of variables to solve the wave equation with the given BC and IC.

Example 25.4. Solve the following partial differential equation (PDE) with boundary and initial conditions (BC & IC). That is, find a function $y(x, t)$ that satisfies the following.

- WE: $y_{tt} = 9y_{xx}$, for $0 \le x \le \pi$ and $t > 0$
- BC: $y(0,t) = 0$ and $y(\pi,t) = 0$, for $t > 0$
- IC: $y(x,0) = x(\pi - x)$ and $y_t(x,0) = 0$, for $0 \le x \le \pi$.

Solution: Step 1. Look for separated solution: $y(x, t) = X(x)T(t)$ to the PDE.

Substitution into WE: $XT'' = 9X''T$.

Algebra: $X''(x)/X(x) = T''(t)/(9T(t)) = \text{ constant } = -\lambda$.

More algebra: $X'' + \lambda X = 0$, $T'' + 9\lambda T = 0$. There are three cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = c\cos(3\sqrt{\lambda}t) + d\sin(3\sqrt{\lambda}t)$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c + dt$.

Case (iii) $\lambda < 0$. Always ignore, since this case only gives the trivial solution satisfying the PDE and boundary conditions.

Step 2. Find which of the solutions in Step 1 also satisfy the boundary conditions $X(0) = 0$, $X(\pi) = 0$.

Case (i) $X(0) = a = 0$ and $X(\pi) = a\cos(\sqrt{\lambda}\pi) + b\sin(\sqrt{\lambda}\pi) = 0$. This has nontrivial solutions when $a = 0$ and $\sqrt{\lambda}\pi = n\pi$ for n an integer. So, in this case, the nontrivial solutions to the PDE satisfying the BC are

$$y_n(x,t) = \sin(nx)(c_n\cos(3nt) + d_n\sin(3nt)) \qquad \text{for } n = 1, 2, 3, \ldots$$

Case (ii) $X(0) = a = 0$, $X(\pi) = a + b\pi = 0$ has only the trivial solution.

Case (iii) Ignored –only has the trivial solution.

Step 3. Both WE and BC are homogeneous, so by superposition the general solution satisfying both is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin(nx)(c_n\cos(3nt) + d_n\sin(3nt)).}$$

Step 4. Use the initial conditions to find the coefficients.

First IC: $y(x,0) = \sum_{n=1} c_n\sin(nx) = x(\pi - x)$. This is the Fourier sine series for $x(\pi - x)$. Now (see the computation section below) the coefficients are

$$\boxed{c_n = \begin{cases} 8/(\pi n)^3 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}}$$

Second IC: $y_t(x,0) = \sum_{n=1} \sin(nx) 3nd_n = 0$. This means that $3nd_n$ are the Fourier sine coefficients for $g(x) = 0$. That is, $\boxed{d_n = 0 \text{ for all } n.}$

We can leave our answer as a set of boxes or put them together in one box.

$$\boxed{y(x,t) = \sum_{n \text{ odd}} \frac{8}{\pi n^3}\sin(nx)\cos(3nt).}$$

25.10 General initial conditions

Example 25.5. Suppose we have the same PDE and BC as in the above example, but the IC are $y(x,0) = f(x)$, $y_t(x,0) = g(x)$. Solve for $y(x,t)$ in terms of the Fourier sine and cosine series f and g.

Solution: Note: since $f(x)$ and $g(x)$ are not specified, the best we can hope to do is give the solution in terms of them.

Since the partial differential equation and boundary conditions are the same, we get the same general solution

$$\boxed{y(x,t) = \sum_{n=1} y_n(x,t) = \sum_{n=1} \sin(nx) \cdot (c_n \cos(3nt) + d_n \sin(3nt)).}$$

First IC: $y(x,0) = \sum_{n=1} c_n \sin(nx) = f(x)$. Therefore, c_n are the Fourier sine coefficients of $f(x)$ on $[0,\pi]$. That is, $\boxed{c_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx}$.

Second IC: $y_t(x,0) = \sum 3n\, d_n \sin(nx) = g(x)$. Therefore, $3n\, d_n$ are the Fourier sine coefficients of $g(x)$ on $[0,\pi]$. That is,

$$3nd_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx \qquad \text{or} \qquad \boxed{d_n = \frac{2}{3n\pi}\int_0^\pi g(x)\sin(nx)\,dx.}$$

25.11 Appendix: Fourier sine and cosine coefficients of $x(L-x)$

We sketch the computation for the Fourier sine and cosine coefficients of $x(L-x)$. The actual integrals can be done by parts or by inspection.

Sine coefficients.

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{2}{L}\int_0^L xL\sin\left(\frac{n\pi}{L}x\right) - x^2\sin\left(\frac{n\pi}{L}x\right)\,dx \\
&= \frac{2}{L}\left[-x\frac{L}{n\pi/L}\cos\left(\frac{n\pi}{L}x\right) + \frac{L}{(n\pi/L)^2}\sin\left(\frac{n\pi}{L}x\right) + \right. \\
&\qquad \left. x^2\frac{1}{n\pi/L}\cos\left(\frac{n\pi}{L}x\right) - 2x\frac{1}{(n\pi/L)^2}\sin\left(\frac{n\pi}{L}x\right) - \frac{2}{(n\pi/L)^3}\cos\left(\frac{n\pi}{L}x\right)\right]_0^L \\
&= \frac{2}{L}\left[-\frac{L^2}{n\pi/L}\cos(n\pi) + \frac{L^2}{n\pi/L}\cos(n\pi) - \frac{2}{(n\pi/L)^3}(\cos(n\pi)-1)\right] \\
&= \begin{cases} 8L^2/(n\pi)^3 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Cosine coefficients.

$$c_0 = \frac{2}{L}\int_0^L x(L-x)\,dx = \frac{2}{L}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^L = \frac{L^2}{3}.$$

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L x(L-x)\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{2}{L}\int_0^L xL\cos\left(\frac{n\pi}{L}x\right) - x^2\cos\left(\frac{n\pi}{L}x\right)\,dx \\
&= \frac{2}{L}\left[x\frac{L}{n\pi/L}\sin\left(\frac{n\pi}{L}x\right) + \frac{L}{(n\pi/L)^2}\cos\left(\frac{n\pi}{L}x\right) + \right. \\
&\qquad \left. -x^2\frac{1}{n\pi/L}\sin\left(\frac{n\pi}{L}x\right) - 2x\frac{1}{(n\pi/L)^2}\cos\left(\frac{n\pi}{L}x\right) + \frac{2}{(n\pi/L)^3}\sin\left(\frac{n\pi}{L}x\right)\right]_0^L \\
&= \frac{2}{L}\left[\frac{L}{(n\pi/L)^2}(\cos(n\pi) - 1) - \frac{2L}{(n\pi/L)^2}(\cos(n\pi))\right] \\
&= \begin{cases} -4L^2/(n\pi)^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$