ES.1803 Topic 26 Notes

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26 PDEs continued

26.1 Goals

1. Reinforce the goals from Topic 25.

26.2 Introduction

The main goal in this topic is to give one more example of the wave equation. This time with boundary conditions that are different from all our previous examples.

As a bonus we also discuss a different method of solving the wave equation called the d'Alembert solution. This is nice, but it only applies to the undamped wave equation. In contrast, the Fourier method applies to many other systems, including the heat equation and the damped wave equation.

As a further bonus we walk through the ratio of frequencies for various musical intervals.

26.3 An example with different BC

Example 26.1. On a string of length $L = \pi$ find $y(x,t)$ satisfying

WE: $y_{tt} = 9y_{xx}$

BC: $y_x(0,t) = 0, \;\; y_x(\pi,t) = 0$

IC: $y(x,0) = f(x), \;\; y_t(x,0) = 0.$

Solution: Note: these are not clamped end boundary conditions. Rather, it is the first partial in x that is 0 at the boundary.

Step 1. Look for separated solutions $y(x,t) = X(x)T(t)$ to WE.

Substitution of $y(x,t) = X(x)T(t)$ into WE gives $XT'' = 9X''T$.

Algebra: $X''(x)/X(x) = T''(t)/(9T(t)) = \text{ constant } = -\lambda$.

More algebra: $X'' + \lambda X = 0, \;\; T'' + 9\lambda T = 0.$

There are three cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x), \;\; T(t) = c\cos(3\sqrt{\lambda}t) + d\sin(3\sqrt{\lambda}t).$

Case (ii) $\lambda = 0$: $X(x) = a + bx, \;\; T(t) = c + dt.$

Case (iii) $\lambda < 0$. Always ignore since this case only gives the trivial modal solutions.

Step 2. (Modal solutions) Find the separated solutions from Step 1 which also satisfy the boundary conditions.

For separated solutions, the BC are $X'(0) = 0, \;\; X'(\pi) = 0.$

Case (i) BC: $X'(0) = \sqrt{\lambda}b = 0$ and $X'(\pi) = -\sqrt{\lambda}a\sin(\sqrt{\lambda}\pi) = 0.$

This has nontrivial solutions when $b = 0$ and $\sqrt{\lambda} = n$ for n an integer. So, in this case, the nontrivial solutions to the PDE satisfying the BC are

$$y_n(x,t) = \cos(nx)(c_n \cos(3nt) + d_n \sin(3nt)) \qquad \text{for } n = 1, 2, 3, \ldots$$

Case (ii) BC: $X'(0) = b = 0$, $X'(\pi) = b = 0$.

So, $X(x) = a$, $T(t) = c + dt$. The factor of a is redundant, so, in this case, the modal solutions is $y(x,t) = c + dt$. As usual with the constant terms, we write this as

$$y_0(x,t) = \frac{c_0}{2} + \frac{d_0 t}{2}$$

Case (iii) Ignored.

Step 3. Both WE and BC are homogeneous, so by superposition we have

$$y(x,t) = \sum_{n=0} y_n(x,t) = \frac{c_0}{2} + \frac{d_0 t}{2} + \sum_{n=1}^{\infty} \cos(nx) \cdot (c_n \cos(3nt) + d_n \sin(3nt))$$

is a solution to WE and BC.

Step 4. Use the initial conditions to find the coefficients.

First IC: $y(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) = f(x)$. That is, we have the Fourier cosine series for $f(x)$.

$$\boxed{c_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx}, \qquad \boxed{c_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx.}$$

Second IC: $y_t(0) = \frac{d_0}{2} + \sum \cos(nx) 3n d_n \Rightarrow d_n = 0$ for $n = 0, 1, 2, \ldots$

So we have our solution to the system (WE, BC, IC):

$$y(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) \cdot \cos(3nt),$$

where the values of c_n are given above.

26.4 Pluck vs. struck initial conditions

A plucked string is one that is held in a starting position and then let go. It has no initial velocity.

A struck string is one that is initially at equilibrium and is struck by an impulse to set it into motion.

So the initial conditions for the two are:
Plucked string: $y(x,0) = f(x)$, $y_t(x,0) = 0$.

Struck string: $y(x,0) = 0$, $y_t(x,0) = g(x)$.

Example 26.2. (Struck string.) A struck string of length $L = \pi$ satisfies the following system

WE: $y_{tt} = 9y_{xx}$

BC: $y(0,t) = 0$, $y(\pi,t) = 0$ (These are different from the previous example.)

IC: $y(x,0) = 0$, $y_t(x,0) = g(x)$

Find the solution.

Solution: WE and BC are the same as Example 25.4. So the general solution satisfying both WE and BC is

$$y(x,t) = \sum_{n=1} y_n(x,t) = \sum_{n=1} \sin(nx) \cdot (c_n \cos(3nt) + d_n \sin(3nt)).$$

As usual, we use the IC to find the values of the coefficents:

First IC: $y(x,0) = 0 = \sum c_n \sin(nx) \;\Rightarrow\; c_n = 0$. So, $\boxed{y(x,t) = \sum d_n \sin(nx)\sin(3nt).}$

Second IC: $y_t(x,0) = g(x) = \sum 3nd_n \sin(nx)$. Therefore, $3nd_n$ are the Fourier sine coefficients of $g(x)$. So,

$$3nd_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx \quad \text{or} \quad \boxed{d_n = \frac{2}{3n\pi}\int_0^\pi g(x)\sin(nx)\,dx.}$$

The two boxed formulas give a complete solution to the example.

26.5 The d'Alembert solution to the wave equation

This secion is for enrichment. We will not cover it in ES.1803

For the undamped, unforced wave equation there is another standard method of solution called the d'Alembert solution. We'll state it and then show how it equals the solution found by the Fourier method.

Consider the system for a plucked string of length L:

WE: $y_{tt} = a^2 y_{xx}$

BC: $y(0,t) = y(L,t) = 0$

IC: $y(x,0) = f(x)$, $y_t(x,0) = 0$.

Claim. Let $\tilde{f}_o(x)$ be the period $2L$ odd extension of $f(x)$. Then

$$y(x,t) = \frac{\tilde{f}_o(x+at) + \tilde{f}_o(x-at)}{2}$$

is a solution to this system. We call this solution the d'Alembert solution.

Proof. This is trivial to check directly! You should do it, and make sure you see why the BC are satisfied.

Note. Physically, we can think of $\tilde{f}_o(x+at)$ as a wave traveling to the left at speed a and $\tilde{f}_o(x-at)$ as the same wave traveling to the right. Since the solution $y(x,t)$ models a standing wave, we see that a standing wave on $[0, L]$ is the superposition of two traveling waves!

26.5.1 The d'Alembert and Fourier solutions are the same

This has to be the case, but we will show it using a standard trig identity.

We know the system has Fourier solution:

$$y(x,t) = \sum b_n \sin\left(\frac{\pi}{L}nx\right)\cos\left(\frac{\pi}{L}ant\right), \quad \text{where } f(x) = \sum b_n \sin\left(\frac{\pi}{L}nx\right) \quad \text{on } 0 < x < L$$

Of course the sine series for $f(x)$ is just the Fourier series for $\tilde{f}_o(x)$, i.e., $\tilde{f}_o(x) = \sum b_n \sin\left(\frac{\pi}{L}nx\right)$ for all x.

We need the following trig identity (which we've used multiple times before).

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta)).$$

Now use this identity to rewrite the Fourier solution.

$$\begin{aligned} y(x,t) &= \sum b_n \sin\left(\frac{\pi}{L}nx\right)\cos\left(\frac{\pi}{L}ant\right) \\ &= \frac{1}{2}\sum b_n \sin\left(\frac{\pi}{L}n(x+at)\right) + \sin\left(\frac{\pi}{L}n(x-at)\right) \\ &= \frac{1}{2}(\tilde{f}_o(x+at) + \tilde{f}_o(x-at)). \qquad \text{QED} \end{aligned}$$

26.6 Musical notes

Assume the fundamental note is a C, then we get the following chart. The 'n' column is the harmonic, i.e., $n = 1$ is the first harmonic (or fundamental), $n = 2$ is the second harmonic, etc. The 'ratio' column is the ratio of the frequencies of the harmonic and the previous harmonic, i.e., $n/(n-1)$

n	ratio	note	interval
1		C	fundamental
2	2/1	C	octave
3	3/2	G	fifth
4	4/3	C	fourth
5	5/4	E	third
6	6/5	G	augmented second
7	7/6		ugly harmonic
8	8/7	C	ignore interval because previous one ugly
9	9/8	D	second
10	10/9	E	second

The point of this chart is to show that the frequencies in musical intervals have small whole number ratios. This is the standard way of tuning a musical instrument. It is called 'just-temperament'.

Another method of tuning is called 'equal temperament. Here the 12 half-steps in an octave are all equal. That is, for each step the ratio of the frequencies is $2^{1/12} \approx 1.05946309$. After 12 steps the frequency has doubled which is an octave. We get the following table. For comparison we include a 'just-tempered' scale.

n 1/2 steps	$2^{n/12}$	interval from base	just tuning interval	percent difference
0	1.0	unison	1	0.00%
1	1.059	minor second	16/15	−0.68%
2	1.122	major second	9/8	−0.23%
3	1.189	minor third	6/5	−0.91%
4	1.260	major third	5/4	+0.79%
5	1.335	perfect fourth	4/3	+0.11%
6	1.414	diminished fifth	7/5	+1.02%
7	1.498	perfect fifth	3/2	−0.11%
8	1.587	minor sixth	8/5	−0.79%
9	1.682	major sixth	5/3	+0.90%
10	1.782	minor seventh	16/9	+0.23%
11	1.888	major seventh	15/8	+0.68%
12	2.0	octave	2/1	0.00%

Note. It is impossible to tune a piano so that all major keys are just-tempered. A piano is called 'well-tempered' when the major keys are close enough to just-tempered that they don't sound out of tune.