ES.1803 Topic 27 Notes

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27 Qualitative behavior of linear systems

27.1 Goals

1. Be able to draw the vector field associated to an autonomous system.
2. Be able to draw the phase portrait of any linear, autonomous, second-order system.
3. Be able to use eigenvalues to classify the types of critical points and their dynamic stability.
4. Be able to use the trace-determinant diagram to organize the different types of critical points.

27.2 Introduction

In this topic we are going to look at the qualitative behavior of systems of the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}, \tag{1}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and A is a constant, 2×2 matrix.

This is a system of two first-order DEs, so it is a second-order system. Since A is constant, the system is autonomous (the rate $\mathbf{x}$ changes depends only on $\mathbf{x}$) and time invariant.

Our goal is to sketch portraits of the solutions to these systems that capture their important qualitative features. Similar to what we did with first-order autonomous equations and phase lines, we will use critical points to organize our work.

While this gives us a useful perspective on linear systems, since we already know how to solve these systems, we don't really need it to understand such systems. Our real goal here is to prepare for a qualitative analysis of nonlinear systems. Since we can't usually solve nonlinear systems exactly, we will approximate them by linear systems and then leverage our qualitative understanding of linear systems to get information about nonlinear ones.

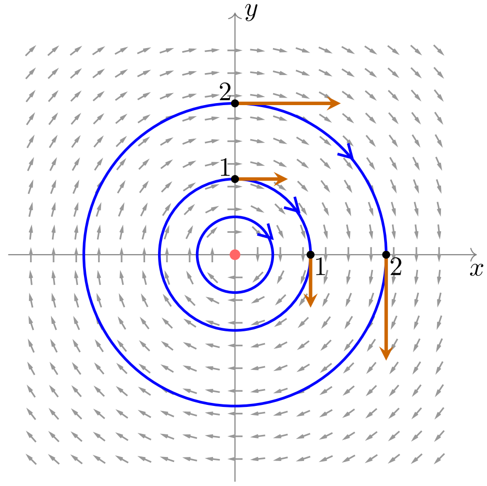
27.3 The phase plane: example with definitions

Example 27.1. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Consider the autonomous system

$$\mathbf{x}' = A\mathbf{x} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2}$$

We'll use this example to define and explain the terms we use in our qualitative description of a system.

Phase plane: The phase plane for our system is simply the xy-plane. This is where we will do all of our graphical work.



Trajectories, tangent vectors and direction field in phase plane for $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Critical points: A critical point of the system is a point in the xy-plane where $\mathbf{x}' = \mathbf{0}$. For the system $\mathbf{x}' = A\mathbf{x}$, critical points satisfy the equation

$$A\mathbf{x} = \mathbf{0}.$$

Every such system has one critical point at $\mathbf{x} = \mathbf{0}$. In our example, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is nonsingular. Therefore, $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$, i.e., the system's only critical point is at the origin. (This is the case for most systems $\mathbf{x}' = A\mathbf{x}$.)

In the phase plane figure above, the critical point at the origin is marked with a solid pink dot.

Trajectories: Any solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ to the system can be plotted as a parametrized curve in the phase plane (xy-plane). Such a curve is called a trajectory of the system.

Using the method of eigenvalues and eigenvectors, we found the solution to Equation 2:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \cos(t) + c_2 \sin(t) \\ -c_1 \sin(t) + c_2 \cos(t) \end{bmatrix} \quad \text{or} \quad x(t) = c_1 \cos(t) + c_2 \sin(t),\ \ y(t) = -c_1 \sin(t) + c_2 \cos(t).$$

Several trajectories are plotted in the figure above. They are circles turning in the clockwise direction.

Important: The constant function $\mathbf{x}(t) = \mathbf{0}$ is a solution to the system. In the figure above, the trajectory of this solution is given by the dot at the origin. That is, the critical point $\mathbf{x} = \mathbf{0}$ is also a stationary trajectory.

Dynamic stability of the equilibrium at the origin: If all solutions go asymptotically to $\mathbf{0}$ as t gets large, we call the equilibrium at the origin a dynamically stable equilibrium. Clearly, the origin is dynamically stable exactly when all the eigenvalues have negative real parts.

If any eigenvalue has a positive real part, then most solutions go to infinity and we call the equilibrium at the origin dynamically unstable.

If the real part of one eigenvalue is 0 and those of all the others are ≤ 0, then we say the equilibrium is an edge case in terms of dynamic stability.

In the example in Equation 2, the eigenvalues are pure imaginary, so this is an edge case. In the figure above, we see the trajectories don't go asymptotically to the origin, but they also don't go to infinity. Whether we consider this stable or not depends on the application we have in mind.

Note: Dynamic stability refers to stability over time. We include the word 'dynamic' to distinguish this type of stability from the notion of structural stability, which we will talk about later.

Vector field and direction field: In general, the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto A \begin{bmatrix} x \\ y \end{bmatrix}$ gives us a vector field in the plane. That is, to each point (x, y) in the plane we attach a vector $A \begin{bmatrix} x \\ y \end{bmatrix}$. The figure above shows these vectors at the points $(1, 0)$, $(2, 0)$, $(0, 1)$, $(0, 2)$. Note: we know the vector field associated with Equation 2 without having to solve the equations.

For a curve $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, the derivative $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ is the tangent or velocity vector. Equation 2 shows that the tangent vectors to trajectories are the same as those in the vector field just described. Notice that the vectors in the figure above at $(1, 0)$, $(2, 0)$, $(0, 1)$, $(0, 2)$ are tangent to the trajectories through these points.

Finally, sometimes, rather than trying to show relative lengths of tangent vector fields, we can make all the vectors the same length. In this case, we call the plot a direction field. It tells you the direction of the trajectory through a point, but not its speed. The figure above shows the direction field (for our system) as a grid of small arrows. Note, at each point on the trajectories, the curve is tangent to the direction field.

27.4 Phase portraits

Definition: To draw the phase portrait of a system of a system, you need to draw enough trajectories to get a good sense of the system. Always include the equilibium solution.

For the remainder of this topic we will consider the general constant coefficient linear system in Equation 1.

This system always has a critical point (i.e., $\mathbf{x}' = 0$) at the origin. A critical point also represents a stationary trajectory, i.e., $\mathbf{x}(t) = \mathbf{0}$ is a solution to Equation 1. Our goal is to use the signs of the eigenvalues to classify the different types of critical points at the origin.

We will divide these types into 'main cases' and 'edge cases'. A main case is one where changing the eigenvalues a little will not change the case. For example, if we have one positive and one negative eigenvalue, then if the eigenvalues change a little, one will remain positive and the other negative.

An edge case is one where the smallest change could change the case. For example, if we have one positive and one zero eigenvalue, then the smallest change in the zero could change this to two positive eigenvalues or one positive and one negative eigenvalue.

Before reading through the cases, you should scan all the phase plane portraits shown below.

27.4.1 Drawing a phase portrait: examples

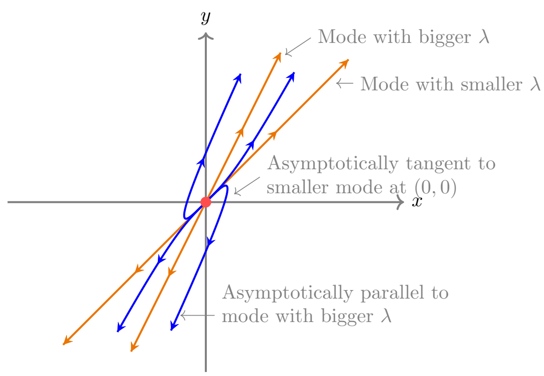
We will use some examples to walk through drawing the phase portrait for several systems. This should be enough to see how to draw phase portraits for all our main cases.

Example 27.2. (Nodal source) Suppose the solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Sketch a phase portrait.

Solution: Here is the final sketch. We outline the steps for drawing the phase portrait below.



Nodal source at $(0, 0)$ – eigenvalues are positive and different.
All trajectories "flow out" from the origin.

Step 1: Sketch the equilibrium solution: $\mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ = single point.

Step 2: Sketch the modes:

Modal solutions: $\mathbf{x_1}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x_2}(t) = c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Mode $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: trajectory = ray from the origin through (1,1).

Mode $\mathbf{x}(t) = -e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: trajectory = ray from the origin through (-1,-1).

Likewise, the trajectories of $\mathbf{x}(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x}(t) = -e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are rays from the origin.

Summary: modes give straight line trajectories.

Step 3: Sketch some "mixed modal" solutions, e.g., sketch $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Asymptotics as $t \to \infty$: Because the eigenvalues (exponents) are positive, as $t \to \infty$, $\mathbf{x}(t)$ goes to infinity. We claim the trajectory becomes asymptotically parallel to the mode with the bigger eigenvalue, i.e., asymptotically parallel to $\begin{bmatrix}1\\2\end{bmatrix}$. To see this, we look at the tangent vector to the trajectory:

$$\mathbf{x}'(t) = 2e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + 3e^{3t}\begin{bmatrix}1\\2\end{bmatrix} = e^{3t}\left(2e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix}\right).$$

This shows that $\mathbf{x}'(t)$ is parallel to $2e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix}$. As t gets large, the first term vanishes and the curve becomes asymptotically parallel to $\begin{bmatrix}1\\2\end{bmatrix}$, as claimed.

Asymptotics as $t \to -\infty$: As $t \to -\infty$, $\mathbf{x}(t)$ goes to zero. We claim the trajectory becomes asympotically tangent to the mode with the smaller eigenvalue, i.e., asymptotically tangent to the line along $\begin{bmatrix}1\\1\end{bmatrix}$. To see this, we look at the tangent vector to the trajectory:

$$\mathbf{x}'(t) = 2e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + 3e^{3t}\begin{bmatrix}1\\2\end{bmatrix} = e^{2t}\left(2\begin{bmatrix}1\\1\end{bmatrix} + 3e^{t}\begin{bmatrix}1\\2\end{bmatrix}\right).$$

This shows that $\mathbf{x}'(t)$ is parallel to $2\begin{bmatrix}1\\1\end{bmatrix}+3e^{t}\begin{bmatrix}1\\2\end{bmatrix}$. So, as t gets large and negative, the second term vanishes and the tangent vector asymptotically points parallel to $\begin{bmatrix}1\\1\end{bmatrix}$, as claimed.

Drawing other mixed modal trajectories is similar.

We call the equilibrium at the origin a nodal source. If you think of the trajectories as representing flowing water, the origin appears as a source, pushing out the water. The equilibrium is dynamically unstable.

Key points

- Trajectories don't cross.
- They fill up the plane.
- Different solutions with the same trajectory have different initial values, e.g.,
  $\mathbf{x_1} = e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{x_2}(t) = 3e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ have the same trajectory, but
  $\mathbf{x_1}(0) = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{x_2}(0) = \begin{bmatrix}3\\3\end{bmatrix}$ are different initial values.
- For nodal sources:
  – Trajectories become parallel to the mode with the bigger λ as t goes to ∞.
  – Trajectories become tangent to the mode with the smaller λ as t goes to $-\infty$.
  – As $t \to -\infty$, trajectories go asymptotically to $(0, 0)$.
  – Systems with positive, different eigenvalues have the same qualitative picture, i.e., they all look like nodal sources.

Example 27.3. (Spiral source) Let $\mathbf{x}' = \begin{bmatrix}3 & 5\\-5 & 3\end{bmatrix}\mathbf{x}$. Draw the phase portrait.

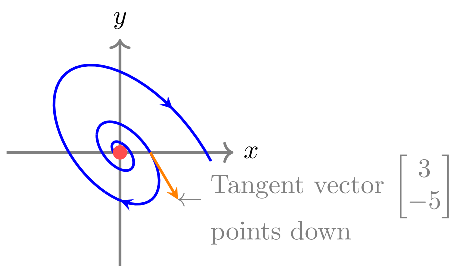
We find the eigenvalues are $\lambda = 3 \pm 5i$. After some algebra we find:

$$\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} \cos(5t) \\ -\sin(5t) \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \sin(5t) \\ \cos(5t) \end{bmatrix}$$

grows × circle = spiral out

To determine the sense of turning, i.e., if it turns clockwise (CW) or counterclockwise (CCW), we look at the tangent vector at the point (1,0) in the plane.

$$\text{At } (1,0): \mathbf{x}'(0) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$
$$= \text{ tangent vector to the trajectory through } (1,0)$$



Phase portrait: spiral source

The tangent vector points down, so the spiral must be turning clockwise.

The critical point at $(0,0)$ is called a spiral source. It is a dynamically unstable equilibrium.

Example 27.4. (Saddle) Suppose the matrix A has the following eigenvalues and eigenvectors.

$$\begin{aligned} \lambda &= \quad -3 \qquad 2 \\ \mathbf{v} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Sketch a phase portrait of the system $\mathbf{x}' = A\mathbf{x}$. Name the type of critical point at the origin and give its stability.

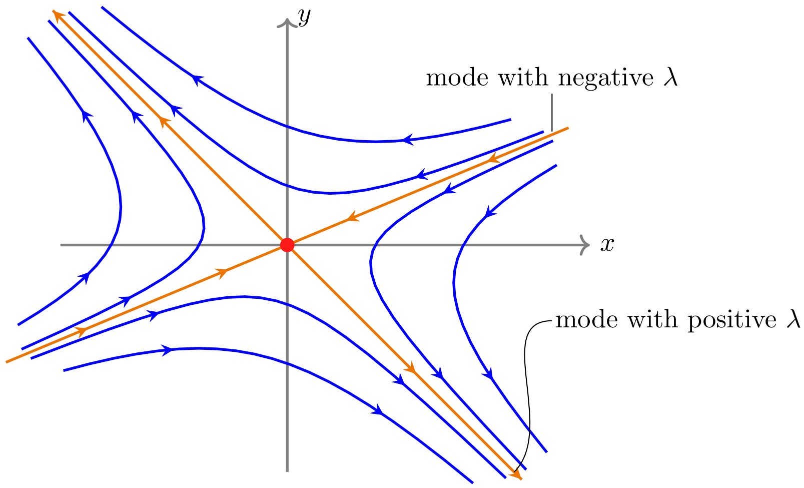
Solution: The general solution is $\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Modes have straight line trajectories:

$$\mathbf{x_1} = e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{goes to 0 as } t \text{ increases.}$$

$$\mathbf{x_2} = e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{goes away from 0 as } t \text{ increases.}$$

Mixed modal solutions: For example, $e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, goes asympotically to $e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as $t \to \infty$ and goes asympotically to $e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as $t \to -\infty$.



Saddle (dynamically unstable equilibrium at (0,0))

27.4.2 Key points about phase portraits

- Trajectories don't cross.
- They fill up the plane.
- Different solutions can have the same trajectory. They just have different initial values.
- Qualitatively, the phase portrait is determined by the eigenvalues.

27.5 Types of critical points: main cases based on eigenvalues

Here we will summarize the main cases for the possible types of critical points (equilibria) at the origin. We'll start with some notational conventions for this section.

If the eigenvalues are real, we label them λ_1 and λ_2. We label the corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. In this case, the general solution to Equation 1 is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2}. \tag{3}$$

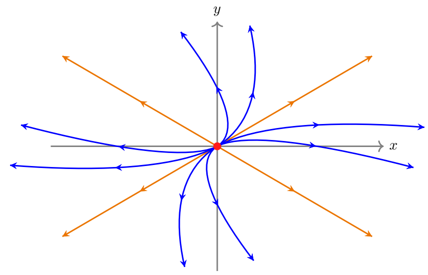
If the eigenvalues are complex (with nonzero imaginary part), we label one of them $\lambda = \alpha + \beta\, i$ and the corresponding eigenvector $\mathbf{v} + i\,\mathbf{w}$. In this case, the general solution to Equation 1 is

$$\mathbf{x}(t) = c_1 e^{\alpha t}(\cos(\beta t)\,\mathbf{v} - \sin(\beta t)\,\mathbf{w}) + c_2 e^{\alpha t}(\sin(\beta t)\,\mathbf{v} + \cos(\beta t)\,\mathbf{w}). \tag{4}$$

Case (i) Real eigenvalues, distinct, both postitive: $\lambda_1 > \lambda_2 > 0$.

Type of critical point at origin: **Nodal source**.

Dynamic stability of the equilibrium: dynamically unstable.



Critical point at the origin is a nodal source

As $t \to \infty$, $\mathbf{x}(t)$ goes to ∞ and the trajectory becomes asymptotically parallel to $\mathbf{v_1}$, i.e., to the eigenvector for the bigger eigenvalue.

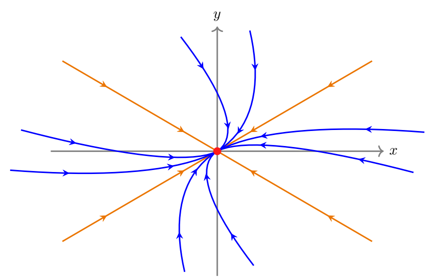
As $t \to -\infty$, $\mathbf{x}(t)$ goes asymptotically to 0 and becomes asymptotically tangent to (the line along) $\mathbf{v_2}$, i.e., to the eigenvector for the smaller eigenvalue.

Case (ii) Real eigenvalues, distinct, both negative, $\lambda_1 < \lambda_2 < 0$.

Type of critical point at origin: **Nodal sink**.

Dynamic stability of the equilibrium: dynamically (asymptotically) stable.

(Simply reverse the arrows on Case (i).)



Critical point at the origin is a nodal sink

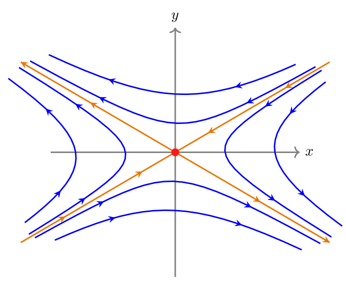
As $t \to \infty$, $\mathbf{x}(t)$ goes asymptotically to 0 and the trajectory becomes asymptotically tangent to (the line along) $\mathbf{v_2}$, i.e., to the eigenvector for the less negative eigenvalue (smaller absolute value).

As $t \to -\infty$, $\mathbf{x}(t)$ goes to ∞ and becomes asymptotically parallel to $\mathbf{v_1}$, i.e., to the eigenvector for the more negative eigenvalue (bigger absolute value).

Case (iii) Real eigenvalues, one positive, one negative, $\lambda_1 > 0 > \lambda_2$.

Type of critical point at origin: **Saddle**

Dynamic stability of the equilibrium: dynamically unstable.



Critical point at the origin is a saddle

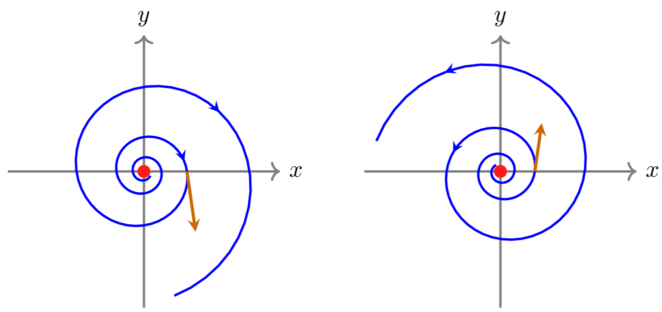
As $t \to \infty$, $\mathbf{x}(t)$ goes to ∞ and becomes asympotically tangent to the mode $c_1 e^{\lambda_1 t}\mathbf{v_1}$, i.e., to the mode with positive eigenvalue.

As $t \to -\infty$, $\mathbf{x}(t)$ goes to ∞ and becomes asympotically tangent to the mode $c_2 e^{\lambda_2 t}\mathbf{v_2}$, i.e., to the mode with negative eigenvalue.

Case (iv) Complex eigenvalues, positive real part, i.e., $\alpha > 0$.

Type of critical point at origin: **Spiral source**

Dynamic stability: dynamically unstable.



Critical point at the origin is a spiral source. Left: clockwise; right: counterclockwise

Trajectories can spiral clockwise or counterclockwise. You can find the direction of rotation by checking the tangent vector at one point.

As $t \to \infty$, $\mathbf{x}(t)$ goes to ∞.

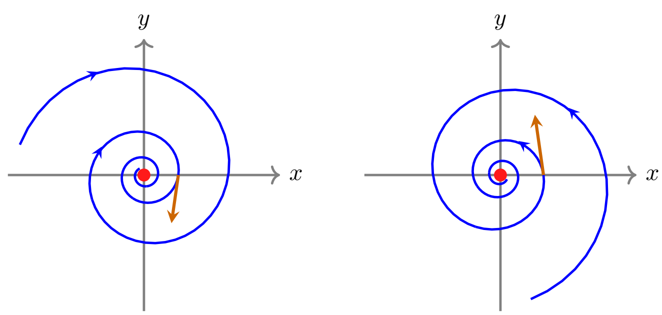
As $t \to -\infty$, $\mathbf{x}(t)$ goes to 0.

Case (v) Complex eigenvalues, negative real part, i.e., $\alpha < 0$.

Type of critical point at origin: **Spiral sink**

Dynamic stability: dynamically stable.

(Reverse arrows from Case (iv).)



Critical point at the origin is a spiral sink. Left: clockwise; right: counterclockwise

Trajectories can spiral clockwise or counterclockwise. You can find the direction of rotation by checking the tangent vector at one point.

As $t \to \infty$, $\mathbf{x}(t)$ goes to 0.

As $t \to -\infty$, $\mathbf{x}(t)$ goes to ∞.

27.6 Types of critical points: edge cases based on eigenvalues

For the edge cases we will just list the properties and show a phase portrait. These are drawn in the same way as the main case examples. In class, we'll look at as many of these as we have time for.

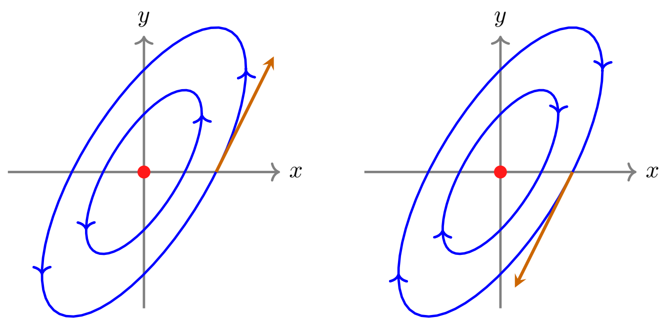
Case (vi) Pure imaginary eigenvalues: $\lambda = i\beta$.

Type of critical point at origin: **Center**

Dynamic stability: This is an edge case, in some applications this can be considered stable, in others it might not.

Trajectories can turn clockwise or counterclockwise. As usual, you can find the direction of rotation by checking the tangent vector at one point.

As $t \to \pm\infty$, $\mathbf{x}(t)$ goes round and round an ellipse.

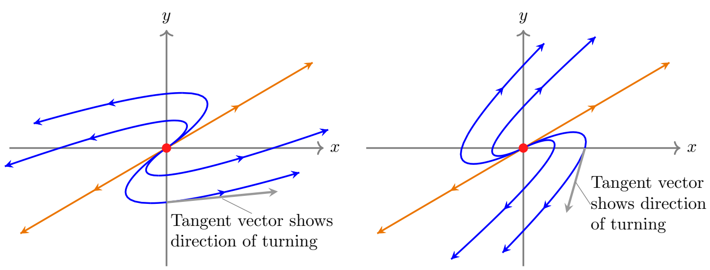


Critical point at the origin is a center

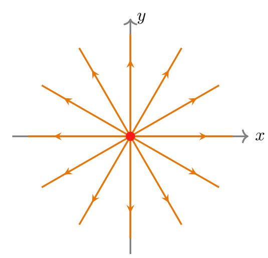
Case (vii) Real, repeated, positive eigenvalues: $\lambda_1 = \lambda_2 > 0$.

Type of critical point at origin: **Defective nodal source or star nodal source.**

Dynamic stability: dynamically unstable .



Defective nodal source



Star nodal source

If the coefficient matrix is **defective** (repeated eigenvalue, only one independent eigenvector), then we have a defective nodal source at the origin.

Let λ be the eigenvalue and $\mathbf{v_1}$ the corresponding eigenvector. Let $\mathbf{v_2}$ be a generalized eigenvector associated with $\mathbf{v_1}$.

In this case, the general solution to Equation 1 is $\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v_1} + c_2(t\mathbf{v_1} + \mathbf{v_2}))$. The critical point at the origin is called a defective nodal source.

As $t \to \infty$, $\mathbf{x}(t)$ goes to ∞ and the trajectory becomes asymptotically parallel to the (only) mode, i.e., parallel to $\mathbf{v_1}$.

As $t \to -\infty$, $\mathbf{x}(t)$ goes to $\mathbf{0}$ and the trajectory becomes asymptotically tangent to the line along $\mathbf{v_1}$.

Trajectories asymptotically make a 180 degree turn. As with spirals, you can find the sense of the turn by checking one tangent vector.

If the coefficient matrix is **complete**, there are two independent eigenvectors, which implies A is a scalar matrix: $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

This implies the general solution is $\mathbf{x}(t) = e^{\lambda t}\vec{\mathbf{c}}$.

That is, all trajectories are straight rays. This is called a star nodal source.

As $t \to \infty$, $\mathbf{x}(t) \to \infty$ along a line from 0.

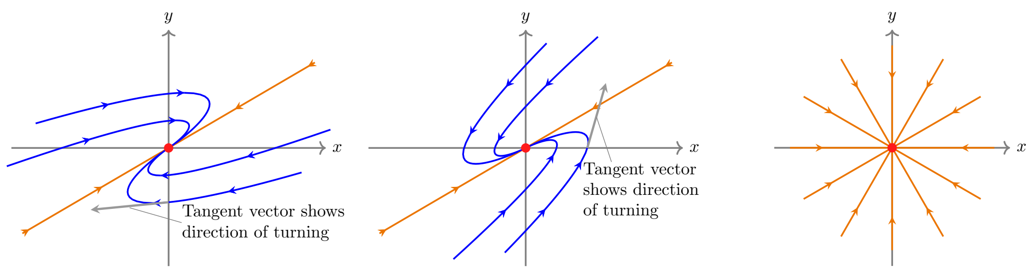
As $t \to -\infty$, $\mathbf{x}(t) \to 0$

Case (viii) Real, repeated, negative eigenvalues: $\lambda_1 = \lambda_2 < 0$.

Type of critical point at origin: **Defective nodal sink or star nodal sink.**

Dynamic stability: dynamically stable.

Just reverse the arrows from Case (vii).



Defective nodal sink Star nodal sink

If the coefficient matrix is defective:

As $t \to -\infty$, $\mathbf{x}(t)$ goes to ∞ and the trajectory becomes asymptotically parallel to the (only) mode, i.e., parallel to $\mathbf{v_1}$.

As $t \to \infty$, $\mathbf{x}(t)$ goes to $\mathbf{0}$ and the trajectory becomes asymptotically tangent to the line along $\mathbf{v_1}$.

Trajectories asymptotically make a 180 degree turn. As with the defective nodal source, you can find the sense of the turn by checking one tangent vector.

If the coefficient matrix is complete, there are two independent eigenvectors, which implies A is a scalar matrix: $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

This implies the general solution is $\mathbf{x}(t) = e^{\lambda t}\vec{\mathbf{c}}$.

(Simply reverse the arrows on the star nodal source.)

Case (ix) Real eigenvalues, one negative, one zero: $\lambda_1 = 0 > \lambda_2$.

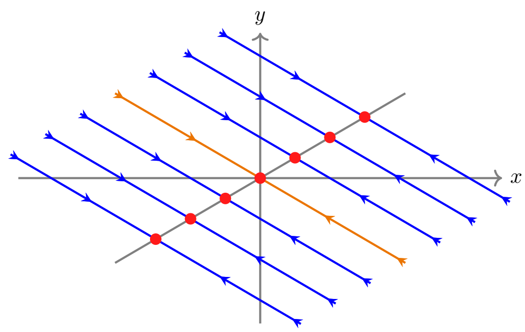
Type of critical point at the origin: **Degenerate** (line of critical points)

Dynamic stability: edge case

The critical points are not isolated –they lie on the line through 0 with direction $\mathbf{v_1}$.

$$\mathbf{x}(t) = c_1\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2}.$$

As $t \to \infty$, $\mathbf{x}(t) \to c_1\mathbf{v_1}$ along a line parallel to $\mathbf{v_2}$.



Degenerate case: line of critical points

Case (x) Real eigenvalues, one positive, one zero: $\lambda_1 = 0 < \lambda_2$.

Type of critical point at origin: **Degenerate** (line of critical points)

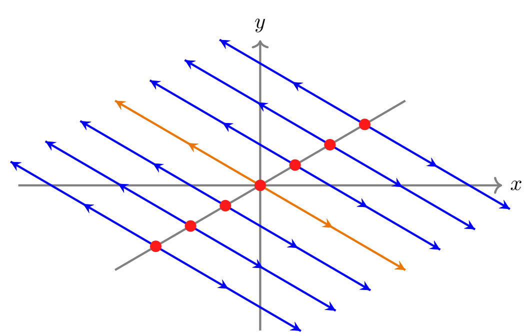
Dynamic stability: dynamically unstable .

(Simply reverse the arrows in Case (ix).)

The critical points are not isolated –they lie on the line through 0 with direction $\mathbf{v_1}$.

$$\mathbf{x}(t) = c_1\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2}.$$

As $t \to \infty$, $\mathbf{x}(t) \to \infty$ along a line parallel to $\mathbf{v_2}$.



Degenerate case: line of critical points

Case (xi) Real eigenvalues, both 0: $\lambda_1 = \lambda_2 = 0$.

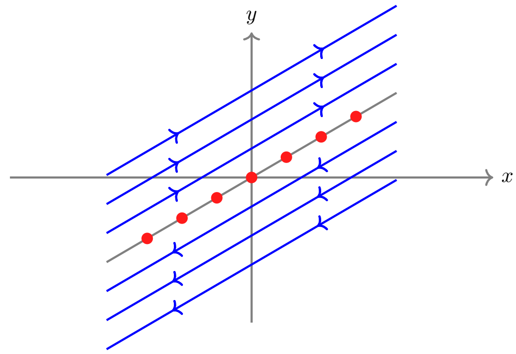
Since the eigenvalues are repeated, this breaks into two cases:

Complete case: Every point is a critical point, every trajectory is a point.

Defective case: Line of critical points.

$$\mathbf{x}(t) = c_1\mathbf{v_1} + c_2(t\mathbf{v_1} + \mathbf{v_2}).$$

Trajectories are parallel to $\mathbf{v_1}$.



Degenerate and defective: (both $\lambda = 0$)

27.7 Example

Example 27.5. The matrix $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ has eigenvalues $2 \pm 3i$. So, for the system $\mathbf{x}' = A\mathbf{x}$, the critical point at the origin is a spiral source.

The tangent vector at the point $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $A\,\mathbf{x}_0 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. This tells us the curve spirals clockwise.

27.8 Trace-determinant plane

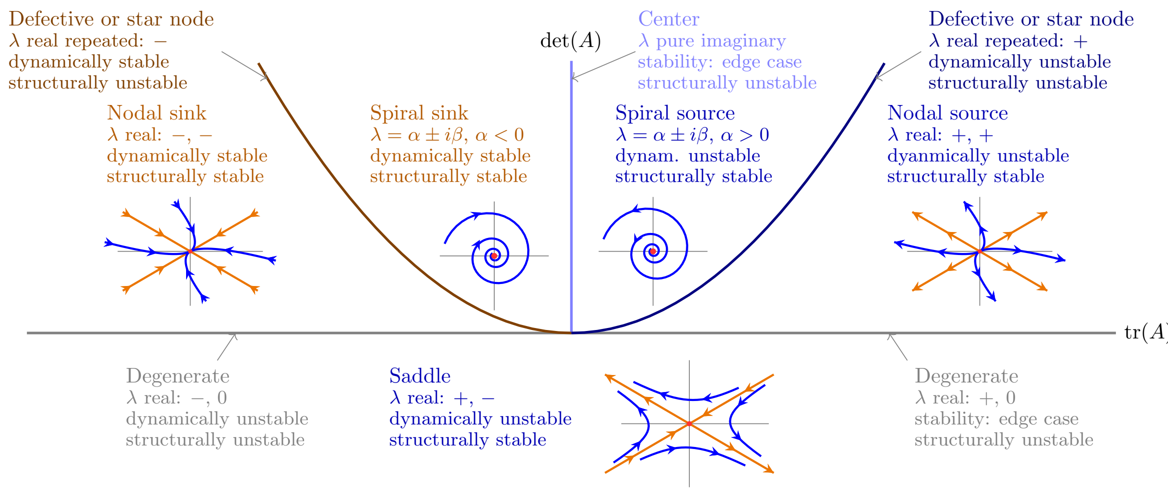
For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

We recognize $ad - bc = \det(A)$. The term $(a+d)$ is called the trace of A, denoted $\mathrm{tr}(A)$. (Trace is the sum of the entries along the main diagonal.) With this notation, the characteristic equation is

$$\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0 \longrightarrow \lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}.$$

Since the eigenvalues are determined by trace and determinant we have the following nice picture in the trace-determinant plane. (Structural stability will be discussed in Topics 28 and 29. To read the diagram, it is enough to know that the main cases are structurally stable and the edge cases are not.)



See the mathlet
https://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/.