# ES.1803 Topic 28 Notes

Jeremy Orloff

## 28 Qualitative behavior of nonlinear systems

### 28.1 Goals

1. Be able to find the critical points for a nonlinear, autonomous system.
2. Be able to linearize a nonlinear system near the critical points.
3. Be able to draw the phase portrait of a nonlinear, autonomous system using linearization near the critical points.
4. Understand why the linearizations in this topic's examples are structurally stable.

### 28.2 Nonlinear Systems

A general first-order, autonomous, $2 \times 2$ system has the following form

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \qquad (1)$$

Vector Field: This defines a vector field $(f(x, y), g(x, y))$ that attaches the velocity vector to each point $(x, y)$ in the *phase plane.*

By definition a critical point is one where $x' = 0$ and $y' = 0$. That is, it is a point $(x_0, y_0)$ where

$$f(x_0, y_0) = 0, \text{ and } g(x_0, y_0) = 0.$$

Equivalently, it is an *equilibrium solution* $x(t) = x_0$, $y(t) = y_0$. This is a solution whose trajectory is a single point.

### 28.3 Approximation and structural stability

We'll talk more about structural stability in Topic 29. The key point is this: if you approximate or measure a number there will be some error. If your approximation says the number is 7, and the error is known to be small, then you can be certain the number's true value is positive. By contrast, if your approximation says the number is 0, then the true value might be positive, negative or zero.

We say a linear system is structurally stable if none of its eigenvalues are 0 or have real part equal to 0. The idea is that, if there is a small change to the system or a small error in our description, then the type of critical point at the origin won't change.

For example, if we experimentally determine a system has eigenvalues 7.0 and 1.0, then our experiment points to the origin being a nodal source. Even if there is a small error in our measurement, we'll still know the eigenvalues are positive and we have a nodal source. We say nodal sources are structurally stable.

In contrast, if we experiemntally find the eigenvalues are $0.0 \pm 2.0\,i$, then our experiment points to the origin being a center. But even the smallest error could mean the eigenvalues have positive or negative real part. That is, all we can say from our experiment is that the origin is a center, spiral source or spiral sink. We say centers are stucturally unstable.

We can state this simply in two ways:

1. The main cases from Topic 27 are structurally stable. The edge cases are not.

2. In the trace-determinant diagram, the large open regions represent structurally stable systems and the dividing lines represent structurally unstable ones.

In this topic we will learn to approximate a nonlinear system near a critical point by a linear one. Because there is approximation error, we can only be sure that the nonlinear system matches the linear one if the linear system is structurally stable. For example, if the linear system is a nodal source, then we can be sure that the nonlinear system looks like a nodal source near the critical point. But, if the linear system is a center, then the nonlinear one could look like a center, spiral source or spiral sink.

All the examples in this topic's notes will involve structurally stable approximations, so we will be confident that we are correctly characterizing the nonlinear system. In Topic 29, we will explore structurally unstable linear approximations.

## 28.4 Linearization around a critical point

We'll start by presenting the method of linearization to sketch the phase portrait. First, we'll use it in an example. After that, we'll justify the method.

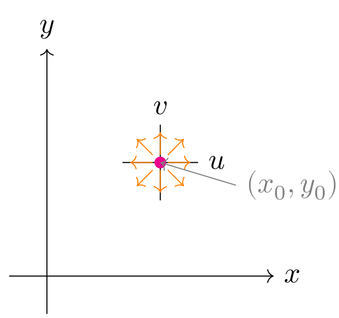
**Jacobian.** At a critical point $(x_0, y_0)$ of the system in Equation 1, we define the Jacobian by

$$J(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}.$$

This gives the linearization around the critical point $(x_0, y_0)$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = J(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}$$

In general, the nonlinear system behaves like the linearized one. (More precisely, if the linearized system is structurally stable, the nonlinear system behaves like the linear one.) That is, if we center our $uv$-axes on $(x_0, y_0)$ then the linear vector field near the $uv$ origin approximates the nonlinear field near $(x_0, y_0)$




Near a critical point, the nonlinear system is approximated by its linearization.

**Example 28.1.** Find the critical points for the following system.

$$x' = 14x - \frac{1}{2}x^2 - xy$$
$$y' = 16y - \frac{1}{2}y^2 - xy$$

**Solution:** We solve the equations $x' = 0$, $y' = 0$.

$$x' = x\left(14 - \frac{1}{2}x - y\right) = 0 \Rightarrow x = 0 \text{ or } 14 - \frac{1}{2}x - y = 0$$
$$y' = y\left(16 - \frac{1}{2}y - x\right) = 0 \Rightarrow y = 0 \text{ or } 16 - \frac{1}{2}y - x = 0.$$

Looking at the product for $x'$ we see $x' = 0$ when $x = 0$ or $14 - x/2 - y = 0$. Likewise, $y' = 0$ when $y = 0$ or $16 - y/2 - x = 0$. This leads to four sets of equations for critical points.

$$\begin{Bmatrix} x = 0 \\ y = 0 \end{Bmatrix} \qquad \begin{Bmatrix} x = 0 \\ 16 - y/2 - x = 0 \end{Bmatrix} \qquad \begin{Bmatrix} 14 - x/2 - y = 0 \\ y = 0 \end{Bmatrix} \qquad \begin{Bmatrix} 14 - x/2 - y = 0 \\ 16 - y/2 - x = 0 \end{Bmatrix}$$

The first three sets are easy to solve by inspection. The fourth requires a small computation. We get the following four critical points:

$$(0,0),\ (0,32),\ (28,0),\ (12,8).$$

**Example 28.2.** (Continued from previous example.) Linearize the system at each of the critical points and determined the type of the linearized critical point.

**Solution:** The linearized system at $(x_0, y_0)$ is $\begin{bmatrix} u' \\ v' \end{bmatrix} = J(x_0, y_0)\begin{bmatrix} u \\ v \end{bmatrix}$.

First we compute the Jacobian:

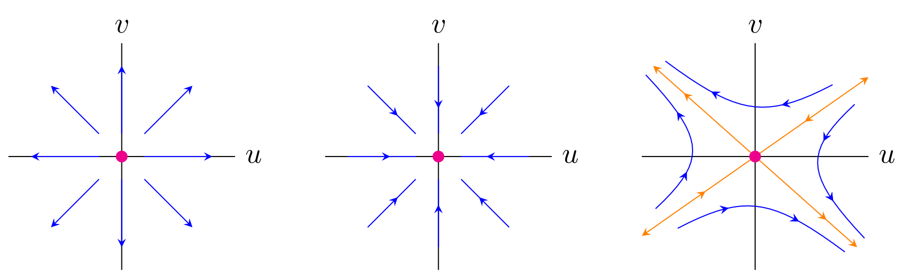
$$J(x,y) = \begin{bmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{bmatrix}$$

Next we look at each of the critical points in turn.

Critical point $(0,0)$:

$$J(0,0) = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix};\ \text{eigenvalues } 14,\ 16.$$

This is a nodal source. Since it is only an approximation of the nonlinear system near the critical point, it is not necessary to find the eigenvectors and make a precise sketch. Instead we draw general nodal source, i.e., a node with all trajectories pointing outward. Its sketch on $uv$-axes is shown in the left-most figure below.




Source node Sink node Saddle

Critical point $(0, 32)$:

$$J(0,32) = \begin{bmatrix} -18 & 0 \\ -32 & -16 \end{bmatrix}; \text{ eigenvalues } -18, -16.$$

This is a sink node. As with the source node, we don't need the eigenvectors to make an approximate sketch of the nonlinear system. We simply sketch a node with all trajectories pointing in towards the critical point. Its sketch is shown in the 'Sink node' figure above.

Critical point $(28, 0)$:

$$J(28,0) = \begin{bmatrix} -14 & -28 \\ 0 & -12 \end{bmatrix}, \text{ eigenvalues } -14, -12; \text{ corresponding eigenvectors } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -14 \\ 1 \end{bmatrix}$$

This is a sink node. As with the source node, we don't need the eigenvectors to make an approximate sketch of the nonlinear system. Its sketch is shown in the 'Sink node' figure above.

Critical point $(12, 8)$:

$$J(12,8) = \begin{bmatrix} -6 & -12 \\ -8 & -4 \end{bmatrix}; \text{ eigenvalues } -5 \pm \sqrt{97} \approx -15, 5.$$

$$\text{Eigenvectors: For } \lambda = -5 - \sqrt{97}: \begin{bmatrix} 1 + \sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$\text{For } \lambda = -5 + \sqrt{97}: \begin{bmatrix} 1 - \sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

This is a saddle. For saddles, we feel it is a good idea to find the eigenvectors so that the orientation of the saddle is correct. (Here, we just gave you the eigenvectors. At this point you should be able to find them quickly yourself.) The sketch of the linearized system is shown in the 'Saddle' figure above.

**Example 28.3.** (Continued from the previous example.) Are all the linearizations structurally stable? What does this imply about the nonlinar system?

**Solution:** Yes. We can see this two ways. First, each of the linearized critical points are one of our main cases. These are structurally stable. Second, all of the eigenvalues for the linearizations are nonzero. Even with a small approximation error, this would still be the case. So the approximation error can't change the types of the critical points, i.e., they are structurally stable.

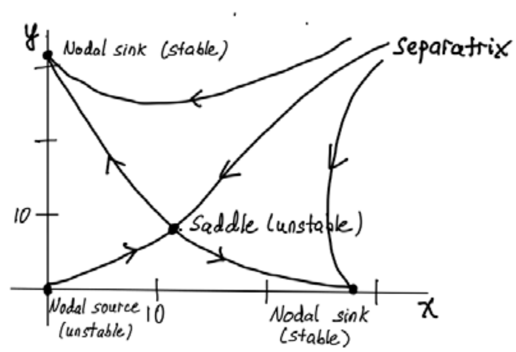
Since all the linearized critical points are structurally stable, the nonlinear critical points are all of the same type as their linearizations.

**Example 28.4.** (Continued from the previous example.) Make a rough sketch of the nonlinear system's phase portrait using the following two steps.

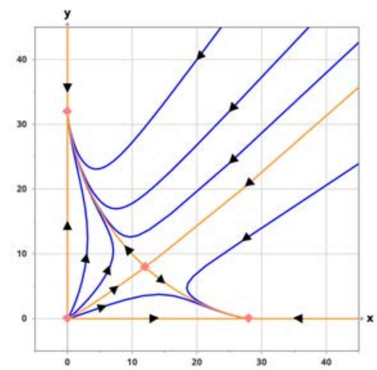
1. Sketch the phase portrait near each critical point, using the linearization.

2. Connect these sketches together in a consistent manner.

We do this below and compare it with a computer generated sketch.



Hand sketch of the phase plane.



Computer generated phase portrait.

### 28.4.1 Justification for using linearization

We'll go through this in detail. One key fact is that the change of variables $u = x - x_0$, $v = y - y_0$ puts the $uv$ origin at $(x_0, y_0)$.

We will use the linear (tangent plane) approximations of $f$ and $g$. You might recall this from 18.02. (If not, notice that it is just a multivariable version of the single variable linear approximation $f(x) \approx f(x_0) + f'(x_0)\Delta x$, where $\Delta x = x - x_0$.)

For small changes $(x - x_0) = \Delta x$ and $(y - y_0) = \Delta y$, the linear approximations for $f$ and $g$ near $(x_0, y_0)$ are

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y$$
$$g(x,y) \approx g(x_0,y_0) + g_x(x_0,y_0)\,\Delta x + g_y(x_0,y_0)\,\Delta y$$

Now, let $u = x - x_0 = \Delta x$ and $v = y - y_0 = \Delta y$.

1. This puts the origin of the $uv$-plane at $(x_0, y_0)$.

3. As functions of $t$: $u' = x'$, $v' = y'$ (since $x_0$ and $y_0$ are constants).

Replacing $x - x_0$ and $y - y_0$ by $u$ and $v$ in the approximations, we get

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)\,u + f_y(x_0,y_0)\,v$$
$$g(x,y) \approx g(x_0,y_0) + g_x(x_0,y_0)\,u + g_y(x_0,y_0)\,v$$

Writing these in matrix form we see the Jacobian appear:

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \approx \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix} + \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix} + J(x_0,y_0) \begin{bmatrix} u \\ v \end{bmatrix}$$

If $(x_0, y_0)$ is a critical point, the first term on the right is 0, i.e

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \approx J(x_0,y_0) \begin{bmatrix} u \\ v \end{bmatrix}.$$

Putting everything together:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \approx J(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}$$

Using just the first and last terms from the above gives the linearization formula

$$\begin{bmatrix} u' \\ v' \end{bmatrix} \approx J(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is a linearized system with coefficient matrix $J(x_0, y_0)$. We call it the linearization of the system around the critical point.