ES.1803 Topic 31 Notes

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31 Applications to physics: mechanical systems

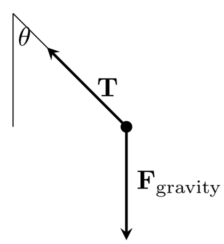
This topic is not officially on the ES.1803 syllabus. It contains several examples of nonlinear physical systems. All of the examples should be accessible to ES.1803 students who have learned through Topic 30.

31.1 Nonlinear pendulum

A pendulum consists of a light rigid rod. It pivots around one end and has a mass m at the other end. Let θ be the (signed) angle the pendulum makes with the vertical direction (see figure). The equation modeling the motion of the pendulum is

$$\theta'' + \frac{g}{l}\sin(\theta) = 0 \quad \text{or} \quad \theta'' + \omega^2 \sin(\theta) = 0,$$

where $\omega^2 = g/l$. (Derivation given below.)



Note: For small θ we can approximate $\theta \approx \sin(\theta)$. With this approximation, the DE becomes $\theta'' + \omega^2\theta = 0$, i.e., for small angles, the nonlinear pendulum is well-approximated by a linear simple harmonic oscillator.

Letting $x = \theta$ and $y = x' = \theta'$, the companion system of the nonlinear equation can be written as

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin(x) \end{aligned}$$

It's easy to establish that the critical points are

$$(n\pi, 0), \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

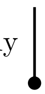
The Jacobian is $J(x,y) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{bmatrix}$.

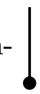
Computing Jacobians and their eigenvalues, we find:

$$n \text{ even} \quad J = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \quad \text{linearized center}$$

$$n \text{ odd} \quad J = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} \quad \text{linearized saddle}$$

Physically, we can describe the equilibria as follows:

n even
(hanging down, dynamically stable) 

n odd
(Pointing up, dynamically unstable) 

31.1.1 Derivation of the pendulum equation

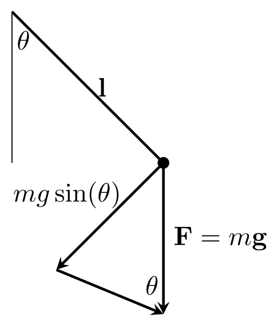
There are many ways to derive this. We do it using rotational mechanics. Energy conservation is another good method.

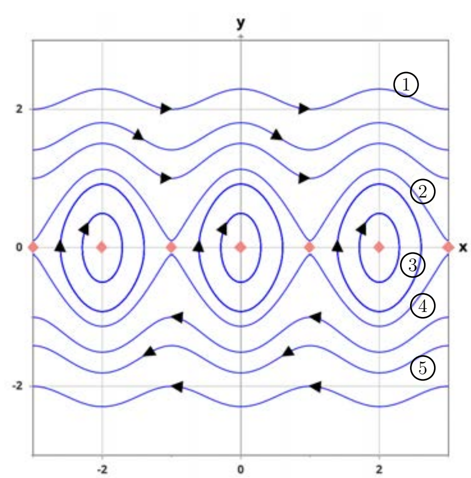
Consider θ to be positive in the counterclockwise direction. So, in the picture, $\theta'' < 0$. We compute the torque about the pivot point.

Torque $= \vec{} = \vec{\mathbf{l}} \times \mathbf{F}_{\text{gravity}}$ has magnitude $lmg \sin\theta$ and points straight down into the page.

We also know that $|\vec{}| = -m^2\,\theta''$. (The minus sign is because $\theta'' < 0$).

This implies $lmg \sin\theta = -m^2\,\theta'' \;\Rightarrow\; \theta'' = -\dfrac{g}{l}\sin\theta$. QED





The labeled trajectories represent:
1. Round and round in a clockwise direction.
2. Just enough energy to asymptotically to the unstable equilibrium.
3. Back and forth (like a, well, pendulum).
4. Like (2) in the opposite direction.
5. Like (1) in the opposite direction.

There are also the equilibria –solid pink dots on the plot;

(6) Marginally stable (centers). (*unlabeled*)
(7) Unstable (saddles). (*unlabeled*)

Note:
The following useful trick allows us to solve for the trajectories exactly.

$$\frac{dy}{dx} = \frac{y'}{x'} = -\frac{\omega^2 \sin x}{y}.$$

This is separable and leads to $y\,dy = -\omega^2 \sin x\,dx$.

Integrating both sides: $\dfrac{y^2}{2} = \omega^2 \cos x + E \ \Rightarrow \ \dfrac{y^2}{2} - \omega^2 \cos x = E$.

We use E as the constant of integration to stand for energy, since this is the usual conservation of total energy equation.

We see that the motion of the pendulum depends on its total energy. We give the possibilities in the following list.

1. $E > w^2$: Trajectory is round and round (trajectories 1, 5).
2. $-\omega^2 < E < \omega^2$: Trajectory is back and forth (trajectory 3).
3. $E = \omega^2$: At or asymptotically approaching the unstable equilibrium (trajectories 2, 4, 7).
4. $E = -\omega^2$: At the stable equilibrium (trajectory 6).
5. $E < -\omega^2$: No trajectory.

31.2 Damped nonlinear pendulum

We can add damping to the pendulum:

$$\theta'' + b\theta' + \omega^2 \sin\theta = 0.$$

The companion system with $x = \theta$, $y = x' = \theta'$ is

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin x - by. \end{aligned}$$

As before, the critical points are at $(n\pi, 0)$ for any integer n.

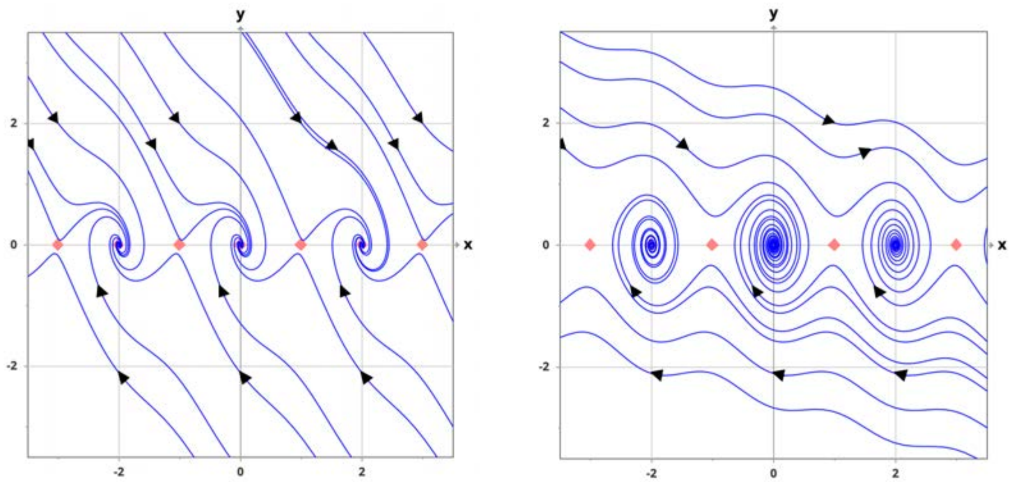
$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{bmatrix} \Rightarrow \begin{cases} n \text{ even} \quad J = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -b \end{bmatrix} & \text{linearized sink} \\ n \text{ odd} \quad J = \begin{bmatrix} 0 & 1 \\ \omega^2 & -b \end{bmatrix} & \text{linearized saddle} \end{cases}$$

The type of linearized sink depends on the sign of the discriminant:

$b^2 - 4\omega^2 < 0 \Rightarrow$ spiral sink

$b^2 - 4\omega^2 > 0 \Rightarrow$ nodal sink

The pictures below show two underdamped nonlinear pendulums.



Damped pendulum Lightly damped pendulum

31.3 Nonlinear Spring

If we add a cubic term to Hooke's law, we get a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \qquad \begin{cases} \textbf{hard if } c < 0 & \text{(cubic term adds to linear force)} \\ \textbf{soft if } c > 0 & \text{(cubic term opposes linear force).} \end{cases}$$

The companion system for these equations is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m \end{aligned}$$

Example 31.1. Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)

Solution: Case 1. Hard spring ($c < 0$): One critical point at $(0, 0)$

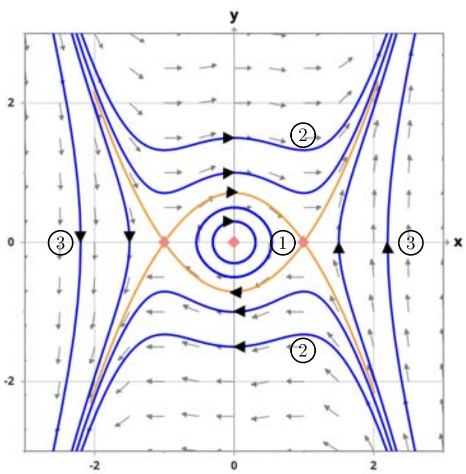
The Jacobian $J(x, y) = \begin{bmatrix} 0 & 1 \\ -k/m + 3cx^2/m & 0 \end{bmatrix}$

$J(0, 0) = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \Rightarrow \lambda = i\sqrt{k/m}$. So we have a linearized center. The problem statement tells us that this is also a nonlinear center.

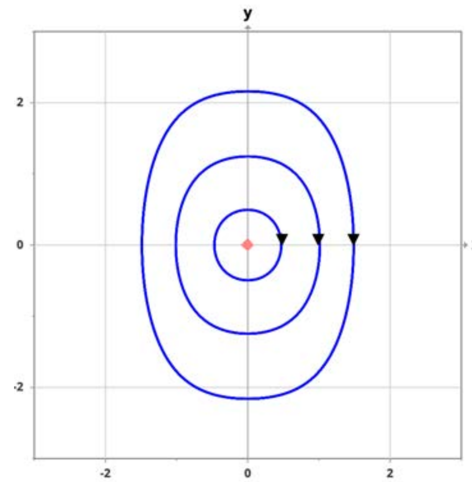
Case 2. Soft spring ($c > 0$): We have the following critical points: $(0, 0)$, $(\pm\sqrt{k/c}, 0)$.

$(0, 0)$: $J(0, 0)$ is the same as for the hard spring. This is a linearized center. The problem statement says it is also a nonlinear center.

$(\pm\sqrt{k/c}, 0)$: $J(\pm\sqrt{k/c}, 0) = \begin{bmatrix} 0 & 1 \\ 2k/m & 0 \end{bmatrix}$ (same for both). Thus we have linearized saddles and, by structural stability, nonlinear saddles. (You should find the eigenvectors to aid in sketching the phase portrait.)



Soft spring: $c > 0$



Hard spring: $c < 0$

Example 31.2. ((Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.

Solution: We use a standard trick to get trajectories:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-kx + cx^3}{my}.$$

This is separable: $my\,dy = (-kx + cx^3)\,dy$. Integrating we get

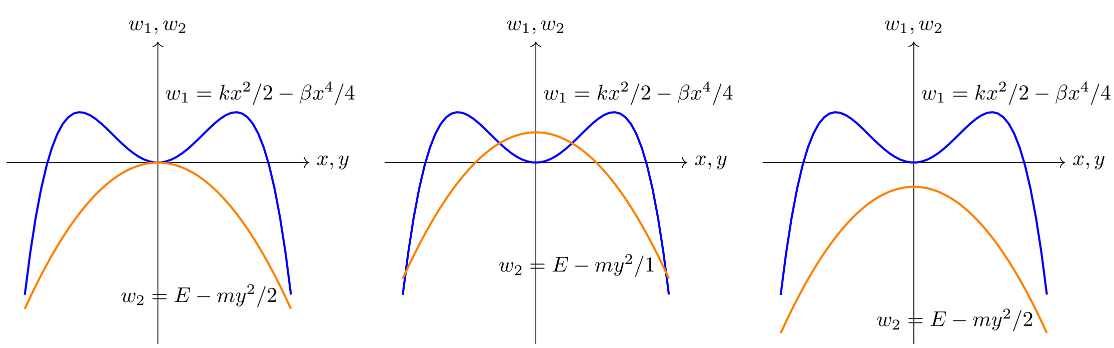
$$\underbrace{\frac{my^2}{2}}_{\text{kinetic energy}} + \underbrace{\frac{kx^2}{2} - \frac{cx^4}{4}}_{\text{potential energy}} = \underbrace{E}_{\text{total energy = constant}}.$$

If $c < 0$ (hard spring), then both energy terms on the right are positive, so x and y must be bounded. Then, for fixed x, there are at most two points on the trajectory. Thus we must have closed trajectories.

If $c > 0$ (soft spring), then, we can define w_1 and w_2 by

$$w_1(x) = \frac{kx^2}{2} - \frac{cx^4}{4}, \quad w_2(y) = E - \frac{my^2}{2}$$

Using $k > 0$, $m > 0$, we have the graphs of w_1, w_2 given below. Using the same graphical ideas as in the proof in the Topic 30 notes that the Volterra predator-prey equation has closed trajectories, this shows the phase plane for the soft spring is as shown above.



Plots of $w_1 = \frac{kx^2}{2} - \frac{cx^4}{4}$, $w_2 = E - y^2$

Similar to the nonlinear pendulum, for the soft spring, different energy levels correspond to different types of trajectories. At the unstable equilibrium we compute $E = \frac{k^2}{4c}$. We have the following correspondence between energy level and trajector (using the labels on the soft-spring phase portrait above):

$E = 0$: Stable equilibrium.

$0 < E < \dfrac{k^2}{4c}$: Trajectories 1.

$E = \dfrac{k^2}{4c}$: Unstable equilibrium, or a trajectory going asymptotically to or from the unstable equilibrium.

$\dfrac{k^2}{4c} < E$: Trajectories 2.

$E < \dfrac{k^2}{4c}$ (including $E < 0$): Trajectories 3

31.4 Damped nonlinear spring

We can add damping to the nonlinear spring: $m\ddot{x} = -kx + cx^3 - b\dot{x}$. As usual we can convert it to a system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m - by/m \end{aligned}$$

Also as usual, we can do a critical point analysis.

Hard spring ($c < 0$): One critical point at $(0, 0)$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \Rightarrow \lambda = \dfrac{-b \pm \sqrt{b^2 - 4km}}{2m}$. So we have 3 possiblities:

(i) underdamped = linearized spiral sink;

(ii) overdamped = linearized nodal sink;

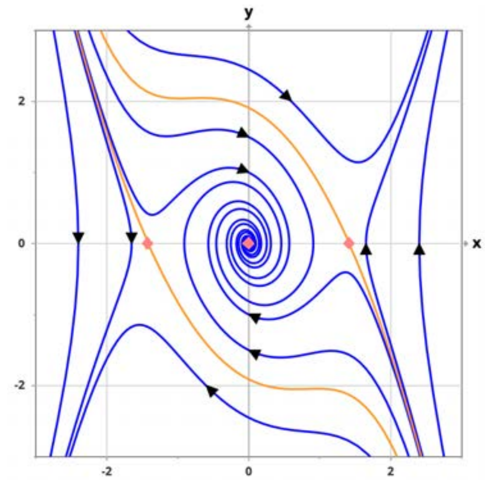
(iii) critically damped = defective sink.

In all cases we have a nonlinear sink. In case (iii), because it's not structurally stable, we would need to do more work to see what type of nonlinear sink we have.

Soft spring ($c > 0$): We have the following critical points: $(0, 0)$, $(\pm\sqrt{k/c}, 0)$.

$(0, 0)$: linearized sink (spiral, nodal or defective), so we have a nonlinear sink.

$(\pm\sqrt{k/c}, 0)$: linearized saddles, so we have nonlinear saddles.


y
2
0
-2
-2
0
2
x