

ES.1803 Topic 4 Notes

Jeremy Orloff

4 Complex numbers and exponentials

4.1 Goals

1. Do arithmetic with complex numbers.
2. Define and compute: magnitude, argument and complex conjugate of a complex number.
3. Be fluent in the use of Euler's formula.
4. Write sine and cosine in terms of complex exponentials ('inverse Euler formulas').
5. Convert complex numbers back and forth between rectangular and polar form.
6. Compute n th roots of complex numbers.

4.2 Motivation

The equation $x^2 = -1$ has no real solutions, yet in ES.1803 we will see that this equation arises naturally and we will want to know its roots. As you may already know, we'll introduce a new symbol for the roots and call it a complex number.

Definition: The symbols $\pm i$ will stand for the solutions to the equation $x^2 = -1$. We will call these new numbers **complex numbers**. We will also write

$$\sqrt{-1} = \pm i$$

Notes: **1.** i is also called an **imaginary number**. This is a historical term. These are perfectly valid numbers that don't happen to lie on the real number line.

2. Our motivation for using complex numbers is not the same as the historical motivation. Mathematicians were willing to say $x^2 = -1$ had no solutions. The problem was in the formula for the roots of cubics. Where square roots of negative numbers appeared even for the real roots of cubics.

3. Engineers typically use j instead of i . We'll follow mathematical custom and use i in ES.1803.

We're going to look at the algebra, geometry and, most important for us, the exponentiation of complex numbers.

Before starting a systematic exposition of complex numbers we'll work a simple example. If the explanation is not immediately clear, it should become clear as we learn more about this topic.

Example 4.1. Solve the equation $r^2 + r + 1 = 0$

Solution: We can apply the quadratic formula to get

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

Think: Do you know how to solve quadratic equations by completing the square? This is how the quadratic formula is derived and is well worth knowing!

4.2.1 Fundamental theorem of algebra

One of the reasons for using complex numbers is because, by allowing complex roots, every polynomial has exactly the expected number of roots.

Fundamental theorem of algebra. A polynomial of degree n has exactly n complex roots (repeated roots are counted with multiplicity.)

Example 4.2. We'll illustrate what we mean by this with a few examples.

1. The polynomial $r^2 + 3r + 2$ factors as $(r + 1)(r + 2)$ therefore its roots are $r = -1$ and $r = -2$. It is a second-order polynomial with 2 roots.
2. The polynomial $r^2 + 6r + 9$ factors as $(r + 3)(r + 3)$. We say it has the roots -3 and -3 . That is it has two roots that happen to be the same. We will also say that -3 is a root of this polynomial with **multiplicity 2**.
3. The polynomial $(r + 1)(r + 2)(r + 3)^2(r^2 + 1)^2$ has degree 8. Its 8 roots are

$$-1, -2, -3, -3, i, i, -i, -i.$$

This example illustrates an important point about polynomials: we prefer to have them in factored form. I think you'll agree that you wouldn't want to find the roots of the polynomial

$$r^8 + 9r^7 + 31r^6 + 57r^5 + 77r^4 + 87r^3 + 65r^2 + 39r + 18.$$

Unless you happened to notice that it was the same as the factored polynomial in Example 4.2(3)! Fortunately, computing packages like Matlab or Octave allow us to find these roots numerically for high order polynomials.

4.3 Terminology and basic arithmetic

Definitions.

- **Complex numbers** are defined as the set of all numbers

$$z = x + yi,$$

where x and y are real numbers.

- We denote the set of all complex numbers by \mathbf{C} . (On the blackboard we will usually write \mathbb{C} —this font is called *blackboard bold*.)

- We call x the **real part** of z . This is denoted by $x = \operatorname{Re}(z)$.
- We call y the **imaginary part** of z . This is denoted by $y = \operatorname{Im}(z)$.

Note well: The imaginary part of z is a real number. It **DOES NOT** include the i .

The basic arithmetic operations follow the standard rules. All you have to remember is that $i^2 = -1$. We will go through these quickly using some simple examples. For ES.1803 it is essential that you become fluent with these manipulations.

- **Addition:** $(3 + 4i) + (7 + 11i) = 10 + 15i$
- **Subtraction:** $(3 + 4i) - (7 + 11i) = -4 - 7i$
- **Multiplication:** $(3 + 4i)(7 + 11i) = 21 + 28i + 33i + 44i^2 = -23 + 61i$. Here we have used the fact that $44i^2 = -44$.

Before talking about division and absolute value we introduce a new operation called conjugation. It will prove useful to have a name and symbol for this, since we will use it frequently.

Complex conjugation is denoted with a bar and defined by

$$\overline{x + iy} = x - iy.$$

If $z = x + iy$ then its conjugate is $\bar{z} = x - iy$ and we read this as “z-bar = $x - iy$ ”.

Example 4.3. $\overline{3 + 5i} = 3 - 5i$.

The following is a very useful property of conjugation. We will use it in the next example to help with division.

Useful property of conjugation: If $z = x + iy$ then $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$.

Example 4.4. (Division.) Write $\frac{3 + 4i}{1 + 2i}$ in the standard form $x + iy$.

Solution: We use the useful property of conjugation to clear the denominator:

$$\frac{3 + 4i}{1 + 2i} = \frac{3 + 4i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{11 - 2i}{5} = \frac{11}{5} - \frac{2}{5}i.$$

In the next section we will discuss the geometry of complex numbers, which give some insight into the meaning of the magnitude of a complex number. For now we just give the definition.

Definition. The **magnitude** of the complex number $x + iy$ is defined as

$$|z| = \sqrt{x^2 + y^2}.$$

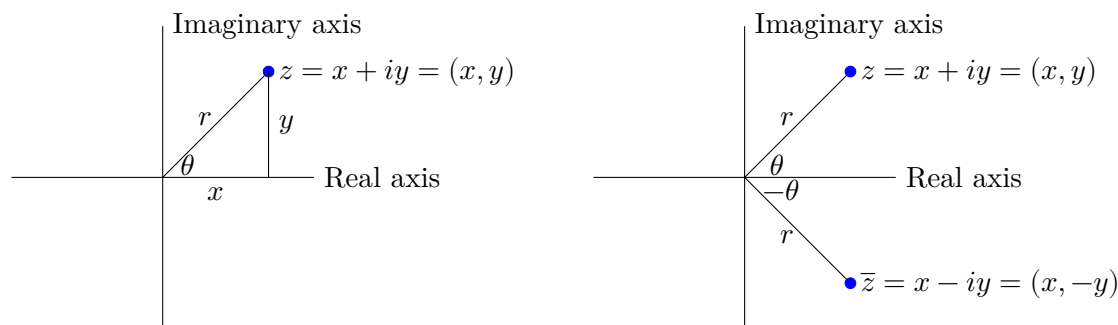
The magnitude is also called the **absolute value** or **norm** or **modulus**.

Example 4.5. The norm of $3 + 5i = \sqrt{9 + 25} = \sqrt{34}$.

Note this really well: The norm is the sum of x^2 and y^2 **it does not include the i !** **Therefore, it is always positive.**

4.4 The complex plane and the geometry of complex numbers

Because it takes two numbers x and y to describe the complex number $z = x + iy$ we can visualize complex numbers as points in the xy -plane. When we do this we call it the **complex plane**. Since x is the real part of z we call the x -axis the **real axis**. Likewise, the y -axis is the **imaginary axis**.



4.5 Polar coordinates

In the figures above we have marked the length r and polar angle θ of the vector from the origin to the point $z = x + iy$. These are the same polar coordinates you saw in 18.02. There are a number of synonyms for both r and θ

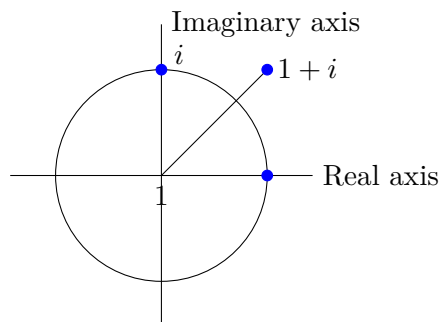
$$r = |z| = \text{magnitude} = \text{length} = \text{norm} = \text{absolute value} = \text{modulus}$$

$$\theta = \text{Arg}(z) = \text{argument of } z = \text{polar angle of } z$$

As in 18.02 you should be able to visualize polar coordinates by thinking about the distance r from the origin and the angle θ with the x -axis.

Example 4.6. In this example we make a table of z , r and θ for some complex numbers. Notice that θ is not uniquely defined since we can always add a multiple of 2π to θ and still be at the same point in the plane.

$z = a + bi$	$r = z $	$\theta = \arg(z)$	
1	1	$0, 2\pi, 4\pi, \dots$	Argument = 0, means z is along the positive x -axis
i	1	$\pi/2, \pi/2 + 2\pi \dots$	Argument = $\pi/2$, means z is along the positive y -axis
$1 + i$	$\sqrt{2}$	$\pi/4, \pi/4 + 2\pi \dots$	Argument = $\pi/4$, means z is along the ray at 45° to the x -axis



4.6 Euler's Formula

Euler's (pronounced 'oilers') formula connects complex exponentials, polar coordinates and sines and cosines. It turns messy trig identities into tidy rules for exponentials. We will use it a lot.

The formula is the following:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta). \quad (1)$$

There are many ways to approach Euler's formula. Our approach is to simply take Equation 1 as the definition of complex exponentials. This is mathematically legal, but does not show that it's a good definition. To do that, we need to show that $e^{i\theta}$ obeys all the rules we expect of an exponential. To do that, we go systematically through the properties of exponentials and check that they hold for complex exponentials.

4.6.1 e^{it} behaves like a true exponential

1. e^{it} differentiates as expected: $\frac{de^{it}}{dt} = ie^{it}$.

Proof. This follows directly from the definition:

$$\frac{de^{it}}{dt} = \frac{d}{dt}(\cos(t) + i \sin(t)) = -\sin(t) + i \cos(t) = i(\cos(t) + i \sin(t)) = ie^{it}.$$

2. $e^{i \cdot 0} = 1$.

Proof. $e^{i \cdot 0} = \cos(0) + i \sin(0) = 1$.

3. The usual rules of exponents hold: $e^{ia}e^{ib} = e^{i(a+b)}$.

Proof. This relies on the cosine and sine addition formulas.

$$\begin{aligned} e^{ia} \cdot e^{ib} &= (\cos(a) + i \sin(a)) \cdot (\cos(b) + i \sin(b)) \\ &= \cos(a) \cos(b) - \sin(a) \sin(b) + i (\cos(a) \sin(b) + \sin(a) \cos(b)) \\ &= \cos(a + b) + i \sin(a + b) = e^{i(a+b)}. \end{aligned}$$

4. The definition of $e^{i\theta}$ is consistent with the power series for e^x .

Proof. To see this we have to recall the power series for e^x , $\cos(x)$ and $\sin(x)$. They are

$$\begin{aligned} e^x &= 1 + x + x^2/2! + x^3/3! + x^4/4! + \dots \\ \cos(x) &= 1 - x^2/2! + x^4/4! - x^6/6! + \dots \\ \sin(x) &= x - x^3/3! + x^5/5! + \dots \end{aligned}$$

Now we can write the power series for $e^{i\theta}$ and then split it into the power series for sine and cosine:

$$\begin{aligned} e^{i\theta} &= \sum_0^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_0^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} + i \sum_0^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\ &= \cos(\theta) + i \sin(\theta). \end{aligned}$$

So the Euler formula definition is consistent with the usual power series for e^z .

1-4 should convince you that $e^{i\theta}$ behaves like an exponential.

4.6.2 Complex exponentials and polar form

Now let's turn to the relation between polar coordinates and complex exponentials.

Suppose $z = x + iy$ has polar coordinates r and θ . That is, we have $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Thus we get the important relationship

$$z = x + iy = r \cos(\theta) + ir \sin(\theta) = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}.$$

This is so important you shouldn't proceed without understanding it. We also record it without the intermediate equation.

$$z = x + iy = re^{i\theta}. \quad (2)$$

Because r and θ are the polar coordinates of (x, y) we call $z = re^{i\theta}$ the **polar form of z** .

4.6.3 Magnitude, argument, conjugate, multiplication and division are easy in polar form

Magnitude. $|e^{i\theta}| = 1$.

Proof. $|e^{i\theta}| = |\cos(\theta) + i \sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$.

In words, this says that $e^{i\theta}$ is always on the unit circle –this is useful to remember!

Likewise, if $z = re^{i\theta}$ then $|z| = r$. You can calculate this, but it should be clear from the definitions: $|z|$ is the distance from z to the origin, which is exactly the same definition as for r .

Argument. If $z = re^{i\theta}$ then $\text{Arg}(z) = \theta$.

Proof. This is again the definition: the argument is the polar angle θ .

Conjugate. $\overline{(re^{i\theta})} = re^{-i\theta}$.

Proof. $\overline{(re^{i\theta})} = \overline{(r(\cos(\theta) + i \sin(\theta)))} = r(\cos(\theta) - i \sin(\theta)) = re^{-i\theta}$.

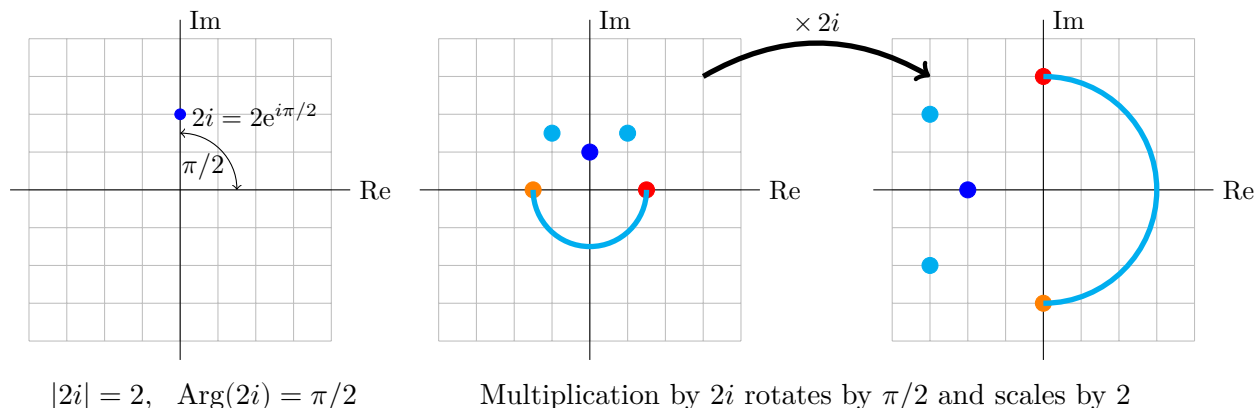
In words: complex conjugation changes the sign of the argument.

Multiplication. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.

This is what mathematicians call trivial to see, just write the multiplication down. In words, the formula says the for $z_1 z_2$ the magnitudes multiply and the arguments add.

Division. Again it's trivial that $\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$.

Example 4.7. Multiplication by $2i$. Here's a simple but important example. By looking at the graph we see that the number $2i$ has magnitude 2 and argument $\pi/2$. So, in polar coordinates, it equals $2e^{i\pi/2}$. This means that multiplication by $2i$ multiplies lengths by 2 and adds $\pi/2$ to arguments, i.e., rotates by 90° . The effect is shown in the figures below



Example 4.8. Raising to a power. Compute (i) $(1 + i)^6$; (ii) $\left(\frac{1+i\sqrt{3}}{2}\right)^3$

Solution: (i) $1 + i$ has magnitude $|1 + i| = \sqrt{2}$ and argument $\text{Arg}(1 + i) = \pi/4$, so $1 + i = \sqrt{2}e^{i\pi/4}$. Raising to a power is now easy:

$$(1 + i)^6 = \left(\sqrt{2}e^{i\pi/4}\right)^6 = 8e^{6i\pi/4} = 8e^{3i\pi/2} = -8i.$$

$$(ii) \frac{1 + i\sqrt{3}}{2} = e^{i\pi/3}, \text{ so } \left(\frac{1 + i\sqrt{3}}{2}\right)^3 = (1 \cdot e^{i\pi/3})^3 = e^{i\pi} = -1$$

4.6.4 Complexification or complex replacement

In the next example we will illustrate the technique of **complexification or complex replacement** by computing a trigonometric integral. Although, in ES.1803, we are not really concerned with trigonometric integrals, we will use complex replacement regularly in other contexts.

Example 4.9. Use complex replacement to compute $I = \int e^x \cos(2x) dx$.

Solution: First we will show the steps for complex replacement. Then, below, we will justify them. We have Euler's formula $e^{2ix} = \cos(2x) + i \sin(2x)$, so $\cos(2x) = \text{Re}(e^{2ix})$. The trick is to replace $\cos(2x)$ by e^{2ix} . We get

$$I_c = \int e^x \cos 2x + ie^x \sin 2x dx, \quad \text{where } I = \text{Re}(I_c).$$

Computing I_c is straightforward:

$$I_c = \int e^x e^{i2x} dx = \int e^{x(1+2i)} dx = \frac{e^{x(1+2i)}}{1+2i}.$$

Now we use polar form to simplify the expression for I_c :

Write $1 + 2i = re^{i\phi}$, where $r = \sqrt{5}$ and $\phi = \text{Arg}(1 + 2i) = \tan^{-1}(2)$ in the first quadrant. Then:

$$I_c = \frac{e^{x(1+2i)}}{\sqrt{5}e^{i\phi}} = \frac{e^x}{\sqrt{5}} e^{i(2x-\phi)} = \frac{e^x}{\sqrt{5}} (\cos(2x - \phi) + i \sin(2x - \phi)).$$

Thus, $I = \operatorname{Re}(I_c) = \frac{e^x}{\sqrt{5}} \cos(2x - \phi)$.

Justification of complex replacement. The trick comes by cleverly adding a new integral to I as follows. Let $J = \int e^x \sin(2x) dx$. Then we let

$$I_c = I + iJ = \int e^x (\cos(2x) + i \sin(2x)) dx = \int e^x e^{2ix} dx.$$

Clearly, $\operatorname{Re}(I_c) = \operatorname{Re}(I + iJ) = I$ as claimed above.

Rectangular coordinates –generally less preferred than polar. We show here the computation in rectangular coordinates –though we hasten to add that in ES.1803 we will almost always prefer polar form because it is easier and gives the answer in a more useable form.

$$\begin{aligned} I_c &= \frac{e^{x(1+2i)}}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{e^x (\cos(2x) + i \sin(2x))(1-2i)}{5} \\ &= \frac{1}{5} e^x (\cos(2x) + 2 \sin(2x) + i(-2 \cos(2x) + \sin(2x))). \end{aligned}$$

So, $I = \operatorname{Re}(I_c) = \frac{1}{5} e^x (\cos(2x) + 2 \sin(2x))$.

4.6.5 Nth roots

We are going to need to be able to find the n th roots of complex numbers. The trick is to recall that a complex number has more than one argument, that is we can always add a multiple of 2π to the argument. For example,

$$2 = 2e^{0i} = 2e^{2\pi i} = 2e^{4\pi i} \dots = 2e^{2n\pi i}$$

Example 4.10. Find all 5 fifth roots of 2.

Solution: In polar form: $(2e^{2n\pi i})^{1/5} = 2^{1/5} e^{2n\pi i/5}$. So the fifth roots of 2 are

$$2^{1/5} = 2^{1/5} e^{2n\pi i/5}, \text{ where } n = 0, 1, 2, \dots$$

The notation is a little strange, because the $2^{1/5}$ on the left side of the equation means the complex roots and the $2^{1/5}$ on the right hand side is a magnitude, so it is the positive real root.

Looking at the right hand side we see that for $n = 5$ we have $2^{1/5} e^{2\pi i}$ which is exactly the same as the root when $n = 0$, i.e., $2^{1/5} e^{0i}$. Likewise $n = 6$ gives exactly the same root as $n = 1$. So we have 5 different roots corresponding to $n = 0, 1, 2, 3, 4$.

$$2^{1/5} = 2^{1/5}, 2^{1/5} e^{2\pi i/5}, 2^{1/5} e^{4\pi i/5}, 2^{1/5} e^{6\pi i/5}, 2^{1/5} e^{8\pi i/5}.$$

Similarly we can say that, in general, $z = r e^{i\theta}$ has N different N th roots:

$$z^{1/N} = r^{1/N} e^{i\theta/N + i2\pi(n/N)} \text{ for } 0, 1, 2, \dots, N-1.$$

Example 4.11. Find the 4 fourth roots of 1.

Solution: $1 = e^{i2\pi n}$, so $1^{1/4} = e^{i2\pi(n/4)}$. So the 4 different fourth roots are 1, $e^{i\pi/2}$, $e^{i\pi}$, $e^{i3\pi/2}$, $e^{i2\pi}$.

When the angles are ones we know about, e.g., 30, 60, 90, 45, etc., we should simplify the complex exponentials. In this case, the roots are 1, i , -1 , $-i$.

Example 4.12. Find the 3 cube roots of -1.

Solution: $-1 = e^{i\pi+i2\pi n}$. So, $(-1)^{1/3} = e^{i\pi/3+i2\pi(n/3)}$ and the 3 cube roots are $e^{i\pi/3}$, $e^{i\pi}$, $e^{i5\pi/3}$. Since $\pi/3$ radians is 60° we can simplify:

$$e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \rightarrow \quad (-1)^{1/3} = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

Example 4.13. Find the 5 fifth roots of $1 + i$.

Solution: $1 + i = \sqrt{2}e^{i(\pi/4+2n\pi)}$, for $n = 0, 1, 2, \dots$. So the 5 fifth roots are

$$(1 + i)^{1/5} = 2^{1/10}e^{i\pi/20}, 2^{1/10}e^{i9\pi/20}, 2^{1/10}e^{i17\pi/20}, 2^{1/10}e^{i25\pi/20}, 2^{1/10}e^{i33\pi/20}.$$

Using a calculator we could write these numerically as $a + bi$, but there is no easy simplification.

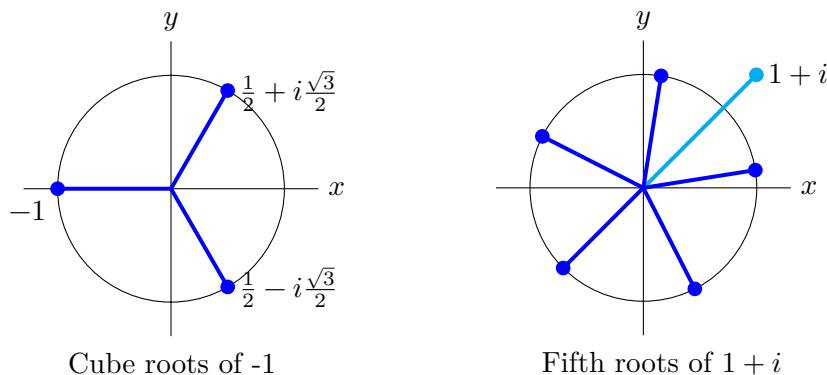
Example 4.14. We should check that our technique works as expected for a simple problem. Find the 2 square roots of 4.

Solution: $4 = 4e^{i2\pi n}$. So, $4^{1/2} = 2e^{i\pi n}$. So the 2 square roots are $2e^0, 2e^{i\pi} = \pm 2$ as expected!

4.6.6 The geometry of n th roots

Looking at the examples above we see that roots are always spaced evenly around a circle centered at the origin. For example, the fifth roots of $1 + i$ are spaced at increments of $2\pi/5$ radians around the circle of radius $2^{1/5}$.

Note also that the roots of real numbers always come in conjugate pairs.



4.7 Inverse Euler Formula

Euler's formula gives a complex exponential in terms of sines and cosines. We can turn this around to get the [inverse Euler formulas](#).

Euler's formula says:

$$e^{it} = \cos(t) + i \sin(t) \quad \text{and} \quad e^{-it} = \cos(t) - i \sin(t).$$

By adding and subtracting we get:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

Warning. We also have the formula $\cos(t) = \operatorname{Re}(e^{it})$ which we used in complex replacement. You want to pay attention to whether this or the inverse Euler formula is appropriate. In general, if you complexified to use complex replacement then at some point you'll need to *decomplexify* by using the formula $\cos(t) = \operatorname{Re}(e^{it})$. If you never complexified then you probably need to use the inverse Euler formula.

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ES.1803 Differential Equations

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