ES.1803 Topic 6 Notes

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6 Operators, inhomogeneous DEs, ERF and SRF

6.1 Goals

1. Be able to define linear differential operators.
2. Be able to define polynomial differential operators and use them to express linear constant coefficient differential equations.
3. Be able to use the Exponential Response Formula to find particular solutions to polynomial differential equations with exponential or sinusoidal input.
4. Be able to derive the Sinusoidal Response Formula.
5. Be able to use the Sinusoidal Response Formula to solve polynomial differential equations with sinusoidal input.
6. Be able to build models of damped harmonic oscillators with input.

6.2 Linear Differential Equations

Linear nth-order differential equations have the form

$$p_0(t)y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \qquad \text{(H)}$$
$$p_0(t)y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f(t) \qquad \text{(I)}$$

As usual, we call (H) homogeneous and (I) inhomogeneous.

Also as usual, if the coefficients are all constant then we have a **constant coefficient** linear differential equation.

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0 \qquad \text{(H)}$$
$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = f(t) \qquad \text{(I)}$$

In Topic 5 we learned about the characteristic equation

$$a_0r^n + a_1r^{n-1} + \cdots + a_n = 0$$

It will be useful to give a name to the polynomial on the left side of this equation.

$$P(r) = a_0r^n + a_1r^{n-1} + \cdot + a_n.$$

We will call it the **characteristic polynomial**. That is, the characteristic equation can be written $P(r) = 0$.

6.3 Operators

A function is a rule that takes a number as input and returns another number as output.

Example 6.1. (Examples of functions.)
1. $f(t) = t^2$. If $t = 2$ is the input then $f(2) = 4$ is the output.
2. The identity function is $f(t) = t$.
3. The zero function is $f(t) = 0$.

An operator is similar to a function except that it takes as input a function and returns another function as output. We will often use upper case letters like T or L to denote operators. If x is a function when T acts on it we will write

$$T(x) \qquad \text{or} \qquad Tx.$$

We will read this as "T of x" or "T applied to x" or "T acting on x." A few examples will make this clear.

Example 6.2. The **differentiation operator** is $D = \frac{d}{dt}$. This takes any function as input and returns its derivative as output. For example,

(i) If $x(t) = t^3$ then $D(x) = 3t^2$. We also write $Dx = 3t^2$.

(ii) If $y(t) = e^{4t}$ then $Dy = 4e^{4t}$.

(iii) $D(t^3 + 2t^2 + 5t + 7) = 3t^2 + 4t + 5$.

(iv) In general, $Dx = x'$.

Example 6.3. The second derivative operator is $D^2 = \frac{d^2}{dt^2}$. For example:

(i) $D^2(e^{4t}) = 4^2e^{4t}$.

In this example we used D^2 to mean first apply D to the function and then apply it again. Writing this out in more detail we get

$$D^2(e^{4t}) = D(D(e^{4t})) = D(4e^{4t}) = 4^2e^{4t}.$$

(ii) In general, $D^2x = x''$. Likewise, $D^3 = x'''$.

For obvious reasons we call D, D^2, D^3, … **differential operators.**

Example 6.4. The **identity operator** I takes any function as input and returns the same function as output. For example:

(i) $I(x) = x$.

(ii) $I(t^2 + 3t + 2) = t^2 + 3t + 2$.

Example 6.5. We can combine these operators. For example we can let

$$T = D^2 + 8D + 7I.$$

To understand what this operator does we have to apply it to a function and see what happens. If we apply T to x we get

$$Tx = (D^2 + 8D + 7I)x = x'' + 8x' + 7x.$$

Example 6.6. The **zero operator** takes any function as input and returns the zero function as output. There is no standard notation for this function, let's call it Z. For example:

(i) $Z(x) = 0$.

(ii) $Z(t^2 + 3t + 2) = 0$.

6.4 Polynomial differential operators

Consider the polynomial $P(r) = r^2+8r+7$. If we replace r by D we have $P(D) = D^2+8D+7$. We will call $P(D)$ a **polynomial differential operator.** We can use it to simplify writing down DEs and to help with algebraic manipulations.

Example 6.7. Consider the constant coefficient differential equation

$$x'' + 8x' + 7x = 0.$$

This has characteristic polynomial $P(r) = r^2+8r+7$. We can rewrite the DE in polynomial notation as

$$(D^2 + 8D + 7I)x = 0 \qquad \text{or, even more simply,} \qquad P(D)x = 0.$$

One great thing about polynomial operators is how simply we can express constant coefficient differential equations using them. We can rewrite (H) and (I) above as

$$P(D) = 0 \tag{H}$$

$$P(D) = f(t), \tag{I}$$

where $P(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n I$.

6.5 Linearity/superposition for polynomial differential operators

The superposition principle was awkward to state and prove because it was phrased in terms of equations. Linearity is equivalent to superposition, but easier to discuss because we phrase it in terms of operators.

Important definition. An operator T is called a **linear operator** if for any functions x_1, x_2 and any constants c_1, c_2 we have

$$T(c_1x_1 + c_2x_2) = c_1Tx_1 + c_2Tx_2. \tag{1}$$

Claim. Show that the differential operator D is linear.

Proof. This is easy to check directly from the definition of linearity:

$$D(c_1x_1 + c_2x_2) = (c_1x_1 + c_2x_2)' = c_1x_1' + c_2x_2' = c_1Dx_1 + c_2Dx_2$$

Looking at the first and last terms in this string of equalities we see that Equation 1 holds for the operator D.

Similarly we can show that the operators D^2, D^3 are linear. Likewise, for any polynomial P, the operator $P(D)$ is linear.

Example 6.8. Show directly from the definition that $P(D) = D^2 + 8D + 7I$ is linear.

Solution: We use the same argument as in the proof of the claim just above:

$$\begin{aligned} P(D)(c_1x_1 + c_2x_2) &= (c_1x_1 + c_2x_2)'' + 8(c_1x_1 + c_2x_2)' + 7(c_1x_1 + c_2x_2) \\ &= c_1(x_1'' + 8x_1' + 7x_1) + c_2(x_2'' + 8x_2' + 7x_2) \\ &= c_1P(D)x_1 + c_2P(D)x_2 \end{aligned}$$

I hope the examples have convinced you that the linearity of an operator is **easy to verify**. You might also have noticed how similar the arguments felt to those showing the superposition principle. For completeness we state and show that the two are equivalent.

Equivalence of linearity and the superposition principle. Suppose T is an operator. Then T is linear if and only if the equation $Tx = q(t)$ satisfies the superposition principle.

Proof. This is really just a matter of unwinding the definitions. Suppose $Tx_1 = q_1$ and $Tx_2 = q_2$. Suppose the superposition principle holds, then

$$T(c_1x_1 + c_2x_2) = c_1q_1 + c_2q_2 = c_1Tx_1 + c_2Tx_2.$$

This shows that T is linear. Likewise, if T is linear, then

$$T(c_1x_1 + c_2x_2) = c_1Tx_1 + c_2Tx_2 = c_1q_1 + c_2q_2,$$

which shows that the superposition principle holds.

6.6 The algebra of $P(D)$ applied to exponentials

For this section $P(D)$ will be a polynomial differential operator and a will be a constant. Here are two easy and useful rules concerning $P(D)$ and e^{ax}. We will use them immediately to show why we have factors of t in the solutions to DEs with repeated roots.

6.6.1 Substitution rule

Substitution rule. $P(D)e^{at} = P(a)e^{at}$. This is called the substitution rule because we just substitute a for D.

'Proof' by example. We show the rule holds for $P(r) = r^2 + 8r + 7$:

$$P(D)e^{at} = (e^{at})'' + 8(e^{at})' + 7e^{at} = (a^2 + 8a + 7)e^{at} = P(a)e^{at}.$$

6.6.2 Exponential shift rule

We will call $P(D + aI)$ a shift of $P(D)$ by a. For example, if $P(D) = D^2 + 6D + 9I$ then

$$P(D - 3I) = (D - 3I)^2 + 6(D - 3I) + 9I = D^2 - 6D + 9I + 6D - 18I + 9I = D^2.$$

Exponential shift rule for D. For any function $u(t)$,

$$D(e^{at}u(t)) = e^{at}\,(D + aI)u(t).$$

Proof. The derivation of this is just the product rule for differentiation:

$$D(e^{at}u(t)) = ae^{at}u(t) + e^{at}u'(t) = e^{at}(au(t) + u'(t)) = e^{at}(D + aI)u(t).$$

Exponential shift rule for D^2. For any function $u(t)$,

$$D^2(e^{at}u(t)) = e^{at}\,(D + aI)^2 u(t).$$

A similar statememt holds for D^3, D^4, ...

Proof. To derive this for D^2 we just use the rule for D twice. Higher powers are similar.

Now it is clear (by linearity!) that the rule applies to any $P(D)$:

Exponential shift rule for $P(D)$. For any function $u(t)$ and polynmial operator $P(D)$,

$$P(D)(e^{at}u(t)) = e^{at}\,P(D + aI)u(t).$$

6.6.3 Repeated roots

We are now in a positition to explain the rule for solutions with repeated roots. Recall:

Rule for repeated roots. If the characteristic equation $P(r)$ has a repeated root r_1 then both $x_1(t) = e^{r_1 t}$ and $x_2(t) = te^{r_1 t}$ are solutions to the homogeneous DE $P(D)x = 0$.

'Proof' by example. Use the exponential shift rule to show the the equation $x'' - 6x' + 9 = 0$ has general solution $x(t) = c_1 e^{3t} + c_2 te^{3t}$.

Solution: First we rewrite this equation in terms of $P(D)$. The characteristic polynomial is

$$P(r) = r^2 - 6r + 9 = (r - 3)^2.$$

So, $P(D) = (D - 3)^2$ and the differential equation is $P(D)x = 0$.

We know $P(r)$ has repeated roots $r = 3, 3$. So, $x(t) = c_1 e^{3t}$ is a solution. Let's vary the parameters to look for other solutions, i.e., let's try $x(t) = e^{3t}u(t)$. We substitute this into the equation and apply the shift rule:

$$\begin{aligned} P(D)x = 0 & \\ &= P(D)(e^{3t}u) \\ &= e^{3t}P(D + 3I)u \\ &= e^{3t}(D + 3I - 3I)^2 u \\ &= e^{3t}D^2 u. \end{aligned}$$

Thus we have the equation $D^2 u = 0$, i.e., $u''(t) = 0$. This is an 18.01 problem and the solution is $u(t) = c_1 + c_2 t$. Putting this back into $x(t)$ we have found

$$x(t) = e^{3t}u(t) = e^{3t}(c_1 + c_2 t),$$

which is exactly what the rule for repeated roots rule said we would find.

6.6.4 Complexification example

Example 6.9. Use complexification to compute $D^3(e^x \sin(x))$.

Solution: We know that $e^x \sin(x) = \text{Im}(e^x e^{ix})$. So, $D^3(e^x \sin(x)) = \text{Im}\left(D^3(e^{x+ix})\right)$. Computing this we have

$$\begin{aligned}(D^3(e^{x+ix})) &= (1+i)^3 e^{x+ix} \\ &= (\sqrt{2}e^{i\pi/4})^3 e^x e^{ix} \\ &= 2^{3/2} e^{i3\pi/4} e^x e^{ix} \\ &= 2^{3/2} e^x e^{i(x+3\pi/4)}\end{aligned}$$

Taking the imaginary part we have

$$D^3(e^x \sin(x)) = \text{Im}\left(D^3(e^{x+ix})\right) = \boxed{2^{3/2} e^x \sin(x + 3\pi/4)}.$$

6.7 Exponential Response Formula

This is one of our key formulas. We will use throughout the rest of ES.1803.

Exponential Response Formula (ERF). Let $P(D)$ be a polynomial differential operator. The inhomogeneous, constant coefficient, linear DE $P(D)y = e^{at}$ has a particular solution

$$y_p(t) = \begin{cases} e^{at}/P(a) & \text{provided } P(a) \neq 0 \\ te^{at}/P'(a) & \text{if } P(a) = 0 \text{ and } P'(a) \neq 0 \\ t^2 e^{at}/P''(a) & \text{if } P(a) = P'(a) = 0 \text{ and } P''(a) \neq 0 \\ \ldots & \ldots \end{cases}$$

Simple proof: The substitution rule says

$$P(D)e^{at} = P(a)e^{at}. \tag{2}$$

If $P(a) \neq 0$, then dividing 2 by $P(a)$ proves the theorem in this case.

If $P(a) = 0$, then we differentiate 2 *with respect to a*. This gives

$$P(D)(te^{at}) = P'(a)e^{at} + P(a)te^{at}.$$

Since $P(a) = 0$, the second term on the right is 0 and we have $P(D)(te^{at}) = P'(a)e^{at}$. Dividing by $P'(a)$ proves the theorem in the case $P(a) = 0$ and $P'(a) \neq 0$.

We can continue in this manner for $P(a) = P'(a) = 0$ etc.

Notes:
1. We will call the cases where $P(a) = 0$ the **Extended Exponential Response Formula.**

2. You will need to know how to use the Extended ERF. You will not be asked to know the proof –although doing so is certainly good for you.

Example 6.10. Let $P(D) = D^2 + 4D + 3I$.

(a) Find a solution to $P(D)x = e^{3t}$.

(b) Find a solution to $P(D)x = e^{-3t}$.

Note: The question only asks for one solution, not all of them.

Solution: (a) The equation has exponential input, so we use the exponential response formula:

Compute, $P(3) = 24$, so the ERF gives $x_p(t) = \dfrac{e^{3t}}{P(3)} = \dfrac{e^{3t}}{24}$.

(b) We try the ERF: Since $P(-3) = 0$, we need the extended ERF.

$P(r) = r^2 + 4r + 3$, so $P'(r) = 2r + 4$ and $P'(-3) = -2$. Thus, $x_p(t) = \dfrac{t\,e^{-3t}}{P'(-3)} = -\dfrac{t\,e^{-3t}}{2}$.

In the next example we combine complex replacement and the ERF.

Example 6.11. Let $P(D) = D^2 + 4D + 5I$. Find a solution to $P(D)x = \cos(2t)$.

Solution: (Long form of the solution with explanatory details.)

First we show the details of replacing $\cos(2t)$ by the complex exponential e^{2it}.

Let $y(t)$ satisfy $P(D)y = \sin(2t)$. Then, by linearity, $z(t) = x(t) + iy(t)$ satisfies the DE

$$P(D)z = P(D)x + iP(D)y = \cos(2t) + i\sin(2t) = e^{2ti} \qquad \text{and } x = \text{Re}(z). \tag{3}$$

Now, in preparation for using the ERF, we compute $P(2i) = 1 + 8i$. Next, we put this in polar form.

$|P(2i)| = |1+8i| = \sqrt{65}$ and $\boxed{\phi = \text{Arg}(P(2i)) = \text{Arg}(1 + 8i) = \tan^{-1}(8) \text{ in quadrant 1.}}$

Thus we have $P(2i) = \sqrt{65}e^{i\phi}$. The ERF gives us complex-valued solution to 3:

$$z_p(t) = \frac{e^{2it}}{P(2i)} = \frac{e^{2it}}{\sqrt{65}e^{i\phi}} = \frac{e^{i(2t-\phi)}}{\sqrt{65}}.$$

All that's left is to take the real part to get a solution to the original DE:

$$x_p(t) = \text{Re}(z_p(t)) = \boxed{\frac{\cos(2t - \phi)}{\sqrt{65}}.}$$

To summarize: $\quad z_p = \dfrac{e^{2i}}{P(2i)}$ and $x_p = \dfrac{1}{|P(2i)|}\cos(2t - \phi)$, where $\phi = \text{Arg}(P(2i))$.

(This example points to the sinusoidal response formula (SRF), which we will look at in the next section.

Example 6.12. Let $P(D) = D^2 + 4D + 5I$. Find a solution to $P(D)x = e^t\cos(2t)$.

Solution: (Short form of solution.) Complexify the DE:

$$P(D)z = e^{-t}e^{2ti} = e^{(-1+2i)t}, \qquad \text{where } x = \text{Re}(z).$$

Side work: $P(-1+2i) = -2+4i = 2\sqrt{5}e^{i\phi}$, where $\boxed{\phi = \text{Arg}(-2 + 4i) = \tan^{-1}(-2), \text{ in Q2}}$.

ERF: $z_p(t) = \dfrac{e^{(-1+2i)t}}{P(-1+2i)} = \dfrac{e^{(-1+2i)t}}{-2+4i} = \dfrac{e^{-t}e^{2it}}{2\sqrt{5}e^{i\phi}} = \dfrac{e^{-t}e^{i(2t-\phi)}}{2\sqrt{5}}.$

Therefore, $x_p = \mathrm{Re}(z_p) = \boxed{\dfrac{e^{-t}}{2\sqrt{5}}\cos(2t-\phi).}$

Example 6.13. With the same $P(D)$ as in the previous example, find a solution to $P(D)x = e^{-2t}\cos(t)$

Solution: Complexify: $P(D)z = e^{-2t}e^{ti} = e^{(-2+i)t}$ where $x = \mathrm{Re}(z)$.

Side work: $P(-2+i) = 0$, so we'll need $P'(-2+i)$:

$P'(r) = 2r+4$, So, $P'(-2+i) = 2i = 2e^{i\pi/2}$.

Extended ERF: $z_p(t) = \dfrac{te^{(-2+i)t}}{P'(-2+i)} = \dfrac{te^{(-2+i)t}}{2e^{i\pi/2}} = \dfrac{te^{-2t}e^{i(t-\pi/2)}}{2}.$

Real part: $x_p(t) = \mathrm{Re}(z(t)) = \boxed{\dfrac{te^{-2t}}{2}\cos(t-\pi/2).}$

You want to get good at this, we will do it a lot.

6.8 The Sinusoidal Response Formula

In the examples above we saw a pattern when the input was sinusoidal. We use it so often that we will codify the result as the Sinusoidal Response Formula.

Sinusoidal Response Formula (SRF). Consider the polynomial differential equation

$$P(D)x = \cos(\omega t)$$

If $P(i\omega) \neq 0$ then the DE has a particular solution

$$x_p(t) = \frac{1}{|P(i\omega)|}\cos(\omega t - \phi(\omega)), \text{ where } \phi(\omega) = \mathrm{Arg}(P(i\omega)).$$

If $P(i\omega) = 0$ we have the **Extended SRF**. For example, if $P(i\omega) = 0$ and $P'(i\omega) \neq 0$ then the DE has a particular solution

$$x_p(t) = \frac{t\cos(\omega t - \phi(\omega))}{|P'(i\omega)|}, \text{ where } \phi(\omega) = \mathrm{Arg}(P'(i\omega)).$$

Proof. To prove the extended SRF we just follow the steps from the examples above.

1. Complexify: $P(D)z = e^{i\omega t}$, where $x = \mathrm{Re}(z)$.
2. Write $P'(i\omega)$ in polar coordinates: $P'(i\omega) = |P'(i\omega)|e^{i\phi(\omega)}$, where $\phi(\omega) = \mathrm{Arg}(P'(i\omega))$.
3. Use the extended ERF: $z_p = \dfrac{te^{i\omega t}}{P'(i\omega)} = \dfrac{te^{i(\omega t-\phi(\omega))}}{|P'(i\omega)|}$.
4. Find the real part of z_p:

$$x_p(t) = \mathrm{Re}(z_p(t) = \mathrm{Re}\left(\frac{te^{i(\omega t-\phi(\omega))}}{|P(i\omega)|}\right) = \frac{t\cos(\omega t-\phi(\omega))}{|P(i\omega)|}.$$

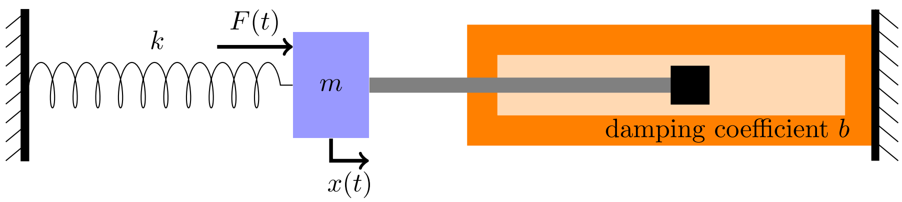
Remember: If in doubt when using the extended SRF, you can always derive it using complexification and the extended ERF.

6.9 Physical models

In this section we will look at three versions of the driven spring-mass-dashpot. These have analogies, which we won't show here, in RLC circuits.

In all three examples, we assume linear damping with damping constant b. That is, if the damper is moving with velocity v through the dashpot, then the force of the dashpot on the damper is $-bv$. This is a reasonable model if the dashpot is filled with a viscous oil.

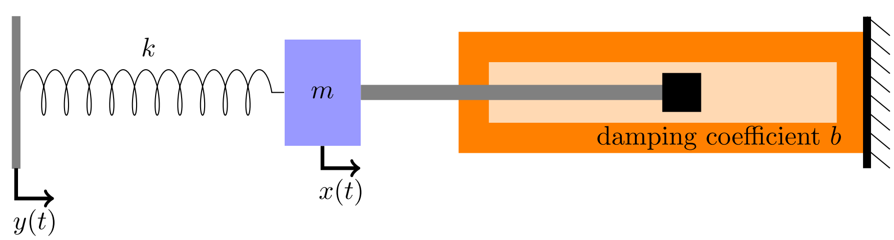
Example 6.14. Driving through the mass. In this version, there is a spring-mass-dashpot which is driven by a variable force applied to the mass as shown. The position of the mass is $x(t)$, with $x = 0$ being the equilibrium position, i.e., the position where the spring is relaxed.



To model this, we consider all the forces on the mass and then use Newton's second law. The spring is stretched by x, so it exerts a restoring force: $-kx$. The velocity of the damper through the dashpot is $\dot{x}$, so it exerts a resisting force: $-b\dot{x}$. Thus Newton's law gives

$$m\ddot{x} = -kx - b\dot{x} + F(t) \;\Leftrightarrow\; \boxed{m\ddot{x} + b\dot{x} + kx = F(t)}.$$

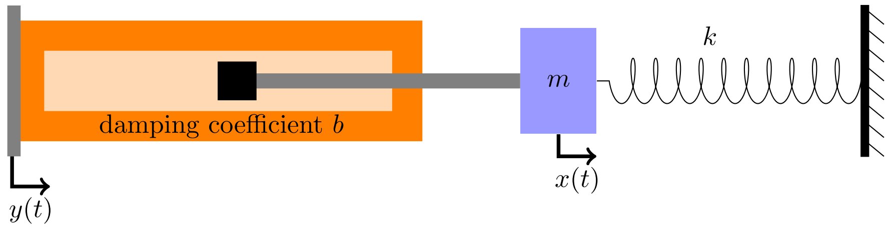
Example 6.15. Driving through the spring. In this version, the spring-mass-dashpot is driven by a mechanism that positions the end of the spring at $y(t)$ as shown. As before, $x(t)$ is position of the mass. We calibrate x and y so that $x = 0$, $y = 0$ is an equilibrium position of the system.



To model this, we must consider all the forces on the mass. At time t, the spring is stretched an amount $x(t) - y(t)$, so the spring force is $-k(x - y)$. Likewise, the velocity of the damper through the dashpot is $\dot{x}$, so the damping force is $-b\dot{x}$. Thus,

$$m\ddot{x} = -k(x - y) - b\dot{x} \;\Leftrightarrow\; \boxed{m\ddot{x} + b\dot{x} + kx = ky}.$$

Example 6.16. Driving through the dashpot. In this version, the spring-mass-dashpot is driven by a mechanism that positions the end of the dashpot at $y(t)$ as shown. Again, $x(t)$ is position of the mass and $x = 0$, $y = 0$ is an equilibrium position of the system.



More briefly than the previous examples:
spring force: $-kx$
damping force: $-b(\dot{x} - \dot{y})$.
Model: $m\ddot{x} = -kx - b(\dot{x} - \dot{y}) \iff \boxed{m\ddot{x} + b\dot{x} + kx = b\dot{y}}$.
