ES.1803 Topic 7 Notes

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7 Solving linear DEs; method of undetermined coefficients

7.1 Goals

1. Be able to solve a linear differential equation by superpositioning a particular solution with the general homogeneous solution.
2. Be able to find a particular solution to a linear constant coefficient differential equation with polynomial input.
3. Be able to work with the operator D, e.g., be able to check if two operators are equal by checking their behavior on test functions.
4. Understand the statement of the existence and uniqueness theorem for second-order linear DEs.

7.2 Linear (not necessarily constant coefficient) DEs

7.2.1 Nice simple operator notation

The general linear differential equation has the form

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f(t) \qquad (1)$$

We can simplify our notation by defining the differential operator

$$L = D^n + p_1(t)D^{n-1} + \cdots + p_n(t)I.$$

Think: In this case, we used the letter L because it is a linear operator. You should recall what this means from Topic 6.

Remember: to see how an operator behaves we apply it to a function. In this case:

$$Lx = \left[D^n + p_1(t)D^{n-1} + \cdots + p_n(t)I\right] x = x^{(n)} + p_1(t)x^{(n-1)} + \cdots p_n(t)x.$$

So we can rewrite Equation 1 as

$$Ly = f(t) \qquad \text{(pretty simple looking).}$$

7.2.2 General solution to a linear inhomogeneous equation

The superposition principle for a linear differential operator L says the following:

- If y_p is a particular solution to the inhomogeneous equation $Ly = f$
- and y_h is a solution to the homogeneous equation $Ly = 0$
- then $y = y_p + y_h$ is also a solution to $Ly = f$.

The proof of this is a straightforward use of the definition of linearity:

$$Ly = L(y_p + y_h) = Ly_p + Ly_h = f + 0 = f.$$

7.2.3 Strategy for finding the general solution to $Ly = f$

1. Find the general solution to the homogeneous equation $Ly = 0$. Call it y_h.
2. Find any one particular solution to $Ly = f$. Call it y_p.
3. The general solution to $Ly = f$ is $y = y_p + y_h$.

Example 7.1. Let $L = D^2 + 4D + 5$. Solve $Ly = e^{-t}$.

Solution: 1. First we solve the homogeneous equation: $Ly = 0$. Since this is a constant coefficient equation, we can use the method of the characteristic equation.

Characteristic equation: $P(r) = r^2 + 4r + 5 = 0$. This has roots $r = -2 \pm i$.

General real-valued homogeneous solution:

$$\boxed{y_h(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t.}$$

2. Find a particular solution using the exponential response formula:

$$\boxed{y_p(t) = \frac{e^{-t}}{P(-1)} = \frac{e^{-t}}{2}.}$$

3. The general real-valued solution to $Ly = e^{-t}$ is a superposition of the particular and the homogeneous solutions:

$$\boxed{y(t) = y_p(t) + y_h(t) = \frac{e^{-t}}{2} + c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t.}$$

7.3 The method of undetermined coefficients for polynomial input

The method of undetermined coefficients for polynomial input is yet another version of the method of optimism. In this case, we try a polynomial solution and use algebra to find the coefficients.

Example 7.2. Solve $y'' + 5y' + 4y = 2t + 3$.

Solution: We follow these steps:

1. First, to find a particular solution:

(a) We guess a trial solution of the form $y_p(t) = At + B$. Our guess has the same degree as the input.

(b) Substitute the guess into the DE and do the algebra to compute the coefficients. Here is one way to present the calculation:

$$\begin{aligned} y_p &= At + B \\ y_p' &= A \\ y_p'' &= 0 \\ y_p'' + 5y_p' + 4y_p &= 5A + 4(At + B) \ = 4At + (5A + 4B) \end{aligned}$$

Substituting this into the DE we get:

$$4At + (5A + 4B) = 2t + 3.$$

Now we equate the coefficients on both sides to get two equations in two unknowns.

$$\begin{array}{llll} \text{Coefficients of } t: & 4A & & = 2 \\ \text{Coefficients of } 1: & 5A & +4B & = 3 \end{array}$$

This is called a triangular system of equations. It is easy: $A = 1/2$, $B = 1/8$. So, $\boxed{y_p(t) = \frac{1}{2}t + \frac{1}{8}.}$

2. Next we find solution of homogeneous DE: $y'' + 5y' + 4y = 0$.

Characteristic equation: $r^2 + 5r + 4 = 0$. This has roots $r = -1, -4$.

General homogeneous solution: $\boxed{y_h(t) = c_1 e^{-t} + c_2 e^{-4t}.}$

3. Finally, we use the superposition principle to write the general solution to our DE:

$$y(t) = y_p(t) + y_h(t) = \frac{1}{2}t + \frac{1}{8} + c_1 e^{-t} + c_2 e^{-4t}.$$

Example 7.3. Solve $y'' + 5y' + 4y = 2t^2 + 3t$.

Solution: Guess a trial solution of the form $y_p(t) = At^2 + Bt + C$ (same degree as the input). Substitute the guess into the DE (we don't show the algebra):

$$y_p'' + 3y_p' + 4y_p = 4At^2 + (10A + 4B)t + (2A + 5B + 4C) = t^2 + 3t.$$

Equate the coefficients of the polynomials on both sides of the equation:

$$\begin{array}{lllll} \text{Coeff. of } t^2: & 4A & & & = 1 \\ \text{Coeff. of } t: & 10A & +4B & & = 3 \\ \text{Coeff. of } 1: & 5A & +5B & +4C & = 0 \end{array}$$

This triangular system is easy to solve: $A = 1/4$, $B = 1/8$, $C = -9/32$. Therefore, a particular solution is

$$\boxed{y_p(t) = \frac{1}{4}t^2 + \frac{1}{8}t - \frac{15}{32}.}$$

We can use the homogeneous solution from previous example. So the general solution to the DE is $y(t) = y_p(t) + y_h(t)$.

Example 7.4. What can go wrong (and how to fix it). Find a solution to

$$y'' + y' = t + 1.$$

Solution: Since the input is a first-degree polynomial, we try a first-degree solution: $y_p(t) = At + B$. Substituting this into the DE we get

$$A = t + 1.$$

This can't be solved!

The problem is that there is no y term in $y'' + y'$ (or rather, its coefficient is 0). The fix is to bump all degrees up by the order of the lowest derivative, i.e., try the solution $y_p = At^2 + Bt$.

Substitute: $2At + (2A + B) = t + 1$.

Equate coefficients: $2A = 1$; $(2A + B) = 1$.

Solve for A and B: $A = 1/2$, $B = 0$.

Thus, $y_p(t) = \frac{1}{2}t^2$.

Example 7.5. Find a solution to $y''' + 3y'' = t$.

Solution: The input has degree 1 and the lowest order derivative in the DE is 2. So we guess $y_p = At^3 + Bt^2$.

Substitute: $18At + 6A + 6B = t$.

Equate coefficients: $18A = 1$; $6A + 6B = 0$.

Solve for A and B: $A = 1/18$, $B = -1/18$.

Thus, $y_p(t) = \dfrac{t^3}{18} - \dfrac{t^2}{18}$.

Example 7.6. Exponential Shift Rule. Solve $y'' + 5y' + 4y = e^{2t}(t + 3)$.

Solution: In operator form this is $P(D)y = e^{2t}(t + 3)$, where $P(D) = D^2 + 5D + 4$.

First we find a particular solution by looking for one of the form $y = e^{2t}u$. We substitute this into the DE and use the exponential shift rule to pull out the exponential. The left-hand side of the equation is

$$P(D)(e^{2t}u) = e^{2t}P(D+2I)u = e^{2t}((D+2I)^2 + 5(D+2I) + 4I)u = e^{2t}(D^2 + 9D + 18I)u.$$

Equating this with the right-hand side we have

$$e^{2t}(D^2 + 9D + 18I)u = e^{2t}(t + 3) \qquad \text{or} \qquad (D^2 + 9D + 18I)u = t + 3.$$

The method of undetermined coefficients gives (we don't show the algebra):

$$u_p(t) = \frac{1}{18}t + \frac{5}{36}.$$

Thus, $\boxed{y_p(t) = e^{2t}u_p(t) = e^{2t}\left(\frac{1}{18}t + \frac{5}{36}\right).}$

To finish solving we find the homogenous solution (again without showing the algebra). $\boxed{y_h(t) = c_1 e^{-t} + c_2 e^{-4t}.}$

So the general solution to the DE is $\boxed{y(t) = y_p(t) + y_h(t).}$

7.4 A bit more on the operator D

We remind you that $D = \frac{d}{dt}$, i.e., $Df = f'$.

7.4.1 Algebra with constant coefficients

For polynomial differential operators, we can add and multiply in any order using the usual rules of arithmetic.

Example 7.7. Show that $(D-3I)(D-2I) = (D-2I)(D-3I) = D^2 - 5D + 6I$.

Note. In words this says that the operators $D-3I$ and $D-2I$ commute and that the usual rules of multiplying polynomials apply.

Solution: To show two operators are equal we have to show they give the same result when applied to any function. This is easy if a bit tedious:

$$(D-3I)(D-2I)f = (D-3I)(f'-2f) = f'' - 3f' - 2f' + 6f = f'' - 5f' + 6f$$
$$(D-2I)(D-3I)f = (D-2I)(f'-3f) = f'' - 2f' - 3f' + 6f = f'' - 5f' + 6f$$

Since $(D-3I)(D-2I)$ and $(D-2I)(D-3I)$ give the same result when applied to a test function f, they are the same operator. The right hand side of both of the equations above shows they both equal $D^2 - 5D + 6I$, as stated in the problem.

The next examples show that we must have constant coefficients for this to work.

7.4.2 Algebra with non-constant coefficient operators

Example 7.8. Let M be the 'multiplication by t' operator, i.e., $Mf = tf$. Show that $MD \neq DM$, i.e., show M and D do not commute.

Solution: We need to apply each operator to a *test function* f and see that we get different results.

$$MDf = Mf' = tf'$$
$$DMf = D(tf) = f + tf'$$

We see that the two are not equal, so the operators don't commute.

Notational note. It is common to use a shorthand and write the operator M as t. So we could have written the example as: Show $tD \neq Dt$ as operators. We will also sometimes write the operator M as tI.

Example 7.9. Show that $(D-tI)D \neq D(D-tI)$.

Solution: As is now usual, we show this by applying both operators to a test function y and seeing that we get different results.

$$(D-tI)Dy = (D-tI)y' = y'' - ty' = (D^2 - tD)y.$$
$$D(D-tI)y = D(y'-ty) = y'' - ty' - y = (D^2 - tD - I)y.$$

7.5 General theory of linear second-order equations

In this section we'll collect up much of what we've already done and add to it the existence and uniqueness theorem.

The general second-order linear DE is

$$A(t)y'' + B(t)y' + C(t)y = F(t).$$

The standard form is

$$y'' + p(t)y' + q(t)y = f(t). \tag{L}$$

Example 7.10. Here is a linear second-order DE in general and standard form:

$$\text{General: } t^2y'' + ty' + y = e^t$$
$$\text{Standard: } y'' + \frac{1}{t}y' + \frac{1}{t^2}\frac{e^t}{t^2} = \frac{e^t}{t^2}$$

Homogeneous (standard form):

$$y'' + p(t)y' + q(t)y = 0 \tag{H}$$

7.5.1 Superposition/Linearity

The general principle of superposition says that, for a linear DE, superposition of inputs leads to superposition of outputs, i.e.

> If y_1 solves $y'' + p(t)y' + q(t)y = f_1(t)$ and y_2 solves $y'' + p(t)y' + q(t)y = f_2(t)$,
> then $c_1y_1 + c_2y_2$ solves $y'' + p(t)y' + q(t)y = c_1f_1(t) + c_2f_2(t)$.

We have already made repeated use of the following two forms of the principle.

1. Superposition of homogeneous solutions: If y_1 and y_2 are solutions to Equation H then so is $y = c_1y_1 + c_2y_2$.

2. Superposition of homogeneous and inhomogeneous solutions: If y_p is a solution to Equation L and y_h is a solution to Equation H then $y = y_p + y_h$ is also a solution to L.

7.5.2 Existence and Uniqueness

The existence and uniqueness theorem is an important technical tool for us. When solving a differential equation it guarantees that we can find a solution and it also tells us when we've found them all.

Theorem. Existence and uniqueness. Consider the initial value problem

$$y'' + p(t)y' + q(t)y = f(t); \qquad y(a) = b_0,\ y'(a) = b_1.$$

If p, q and f are continuous on an interval I containing the point a then there exists a unique solution to this differential equation satisfying the given initial conditions.

Important graphical note: The theorem tells us that the graphs of two different solutions to the DE *can* cross, but they *cannot* touch tangentially.

(i) If they cross transversally, then they have the same position at the time they cross, but different velocities.

(ii) If they did touch tangentially, then they would have the same position and the same velocity at the time they touch. By the existence and uniqueness theorem, there is exactly one solution –not two– with that position and velocity, so this is impossible.

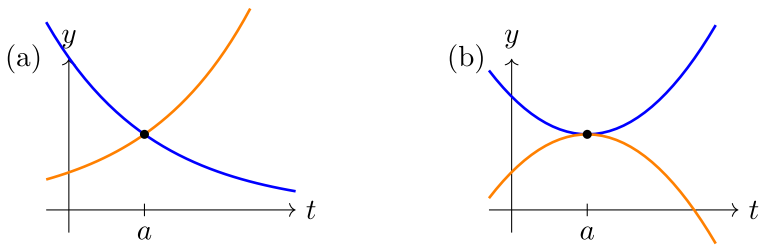

Figure (a) shows transversal curves. These could both be solutions to a second-order DE that satisfies the conditions of the existence and uniqueness theorem.

Figure (b) shows curves that touch tangentially. These cannot both be solutions to such a DE.