

ES.1803 Topic 9 Notes

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9 Applications: frequency response

9.1 Goals

1. Be able to use the engineering terminology of gain, phase lag, resonance.
2. Understand that the gain depends on what we declare to be the input.
3. Be able to find practical or pure resonant frequencies if they exist.

9.2 Review of a forced damped harmonic oscillator

Note. You can also see the text by Edwards and Penney, sections 2.4 and 2.7 for a nice discussion of RLC circuits and practical resonance.

Throughout this topic we will be considering damped harmonic oscillators. There will be important variations, but let's start by reviewing one such system.

Example 9.1. Consider the system

$$my'' + by' + ky = kB \cos(\omega t), \quad (1)$$

where m , b , k , B , and ω are constants. For this system, we will **consider** $B \cos(\omega t)$ to be the **input**. Below we will discuss how the input and output are not mathematical notions. In an engineering context we must always say what we mean by the input and the output.

Let's review our method of solution for this equation

1. Find the homogeneous solution.

$$\text{Characteristic roots} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Let $\beta = \frac{\sqrt{|b^2 - 4mk|}}{2m}$. (Note the absolute value inside the square root.) There are three cases:

- (i) $b^2 - 4mk > 0$ (overdamped): $y_h(t) = c_1 e^{(-b/2m + \beta)t} + c_2 e^{(-b/2m - \beta)t}$.
- (ii) $b^2 - 4mk < 0$ (underdamped): $y_h(t) = c_1 e^{-bt/2m} \cos(\beta t) + c_2 e^{-bt/2m} \sin(\beta t)$.
- (iii) $b^2 - 4mk = 0$ (critically damped): $y_h(t) = c_1 e^{-bt/2m} + c_2 t e^{-bt/2m}$.

2. Find a particular solution.

We can apply the **sinusoidal response formula** (SRF) directly:

$$y_p(t) = \frac{kB \cos(\omega t - \phi(\omega))}{|P(i\omega)|},$$

where $P(r)$ is the characteristic polynomial and $\phi(\omega) = \text{Arg}(P(i\omega))$.

Because we will want to make small variations in this formula we will also review the method of complexification that leads to the sinusoidal response formula.

Step 1. Complexify the DE to get:

$$mz'' + bz' + kz = kB e^{i\omega t}, \quad \text{where } y = \text{Re}(z).$$

Step 2. We will need $P(i\omega)$ in polar form. The characteristic polynomial is $P(r) = mr^2 + br + k$. So,

$$P(i\omega) = (k - m\omega^2) + ib\omega = \sqrt{(k - m\omega^2)^2 + b^2\omega^2} e^{i\phi(\omega)},$$

where $\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$ in the first or second quadrants.

Think: Why is $\phi(\omega)$ in Q1 or Q2?

Step 3. Use the exponential response formula to give a particular (complex-valued) solution:

$$z_p = \frac{kB e^{i\omega t}}{P(i\omega)} = \frac{kB e^{i(\omega t - \phi(\omega))}}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}. \quad (2)$$

Step 4. *Uncomplexify* by taking the real part to find y_p .

$$y_p(t) = \text{Re}(z(t)) = \frac{kB \cos(\omega t - \phi(\omega))}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}. \quad (3)$$

Finally we use superposition to give the general real-valued solution:

$$y(t) = y_p(t) + y_h(t).$$

9.2.1 Terminology

Still referring to the system in Example 9.1:

- $y_h(t)$ is called the **transient** because it goes to 0 as t goes to infinity.

Think: How do we know that $y_h(t)$ decays to 0?

- $y_p(t)$ is called the **periodic** or **sinusoidal solution**.

Since $y_h(t)$ goes to 0, all solutions go asymptotically to $y_p(t)$.

In thinking about this system, we are going to assume the m , b , and k are fixed. We will imagine that we have a knob that can be used to set ω just before we need to solve the equation. Thus the response of the system will depend on the value of ω .

The following is a list of terms with short definitions. We will discuss them in much more detail below.

- **Input:** When talking about gain and phase lag, we will always take the input to be sinusoidal, i.e., $B \cos(\omega t)$.

- **Input frequency:** The angular frequency of the sinusoidal input, i.e., ω . (In radians/time.)
- **Input amplitude:** The amplitude of the sinusoidal input.
- **Output amplitude:** The amplitude of the sinusoidal solution.
- **Gain or amplitude response:** the amount by which the system scales the input amplitude to get the output amplitude, i.e., the **ratio of the output to input amplitudes**.
- **Complex gain:** the ‘gain’ for the complexified equation, i.e., the ratio of output to input.
- **Phase lag:** the angle by which the output maximum trails the input maximum.
- **Time lag:** the time by which the output maximum trails the input maximum.
- **Frequency response:** both amplitude response and phase lag taken together.

By looking at the solutions in Equations 2 and 3, we can give these quantities for the system discussed above. Pay attention to the abstract statements involving $P(i\omega)$, they are more useful to know than the formulas with square roots etc.

- Input: $B \cos(\omega t)$.
- Input frequency: ω .
- Input amplitude: Since we declared the input to be $B \cos(\omega t)$, the input amplitude is B .
- Output amplitude: $A(\omega) = \frac{kB}{|P(i\omega)|} = \frac{kB}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$.
- Gain: The gain is the ratio of the output amplitude to the input amplitude. So the gain $g(\omega)$ is

$$g(\omega) = \frac{kB/|P(i\omega)|}{B} = \frac{k}{|P(i\omega)|} = \frac{k}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}.$$

- Complex gain: In the complexified DE, we replace $i\omega$ by s to get the equation with exponential input:

$$P(D)z = Be^{st},$$

where s can be any complex number, e.g., $i\omega$ or $2 + 3i$.

The input is Be^{st} and the output is $\frac{kBe^{st}}{P(s)}$. The complex gain is the ratio of output to input: $\frac{kBe^{st}/P(s)}{Be^{st}} = \frac{k}{P(s)} = \frac{k}{ms^2 + bs + k}$.

- Phase lag: $\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$ in Q1 or Q2.
- Time lag: $\phi(\omega)/\omega$.

9.2.2 Input and gain

Important note: The gain depends on what we designate as the input. Do not try to memorize the exact formulas for gain in the example above. In other systems the formulas will be slightly different. You will need to think about each system! Pay attention to this in all the examples below.

Example 9.2. Consider the damped harmonic oscillator driven by pushing on the end of the spring. If $f(t)$ is the displacement of the end from its equilibrium position, then the system is modeled by

$$mx'' + bx' + kx = kf(t).$$

In this case, it is reasonable to consider $f(t)$ to be the input.

Taking $f(t) = B \cos(\omega t)$, this is exactly the DE from Example 9.1 above. We saw that this has gain $g(\omega) = \frac{k}{|P(i\omega)|}$.

Example 9.3. Consider the system

$$2y'' + 1.5y' + 3y = 3B \cos(\omega t)$$

where we consider $B \cos(\omega t)$ to be the input. (Note the input does not include the factor of 3). Plot the graph of the gain as a function of ω .

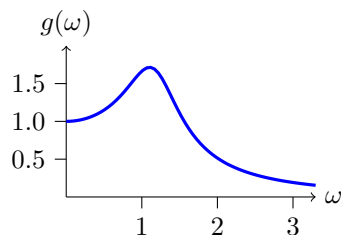
Solution: The sinusoidal solution to this equation is

$$y_p = \frac{3B \cos(\omega t - \phi(\omega))}{|P(i\omega)|} = \frac{3B}{\sqrt{(3 - 2\omega^2)^2 + (1.5\omega)^2}} \cos(\omega t - \phi(\omega)) \quad (\text{where } \phi(\omega) = \text{Arg}(P(i\omega))).$$

So the gain (output amplitude/input amplitude) is

$$g(\omega) = \frac{3}{\sqrt{(3 - 2\omega^2)^2 + (1.5\omega)^2}}.$$

Here is the plot of $g(\omega)$:

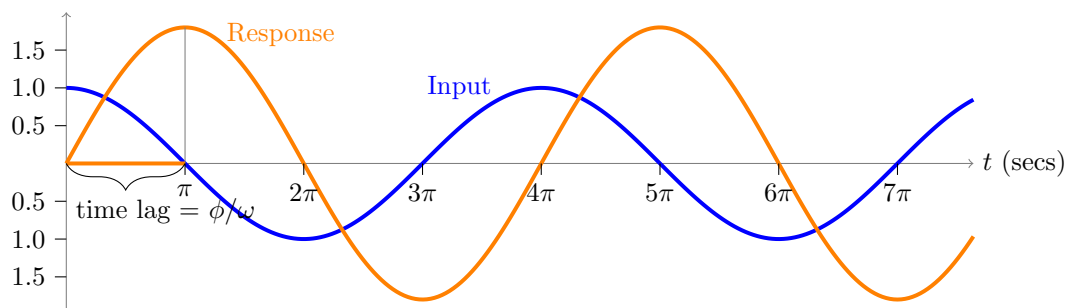


Graph of the gain function for Example 9.3

9.2.3 Phase Lag

Example 9.4. In the figure below the blue curve is the input and the orange curve is the response. The damping causes a lag between the time the input reaches its maximum and the time the output reaches its maximum.

- The figure shows that the output lags π seconds behind the input. This is the **time lag**
- The period of both input and response is 4π seconds. So the output is $\pi/4\pi = 1/4$ cycle = $\pi/2$ radians behind the input. The angle $\phi = \pi/2$ radians is the **phase lag**.



The response lags behind the input by π seconds or $\pi/2$ radians.

The phase lag is important in many applications, but in this class we will be more interested in the gain.

9.3 Amplitude response and practical resonance

The gain is a function of ω . It tells us the size of the system's response at the given input frequency. If the gain has a relative maximum at $\omega_r > 0$, then we call ω_r a **practical resonant frequency**.

Example 9.5. (Finding practical resonance.) Consider the system from Example 9.3:

$$2y'' + 1.5y' + 3y = 3B \cos(\omega t).$$

As in that example, we consider $B \cos(\omega t)$ to be the input. Find all the practical resonant frequencies.

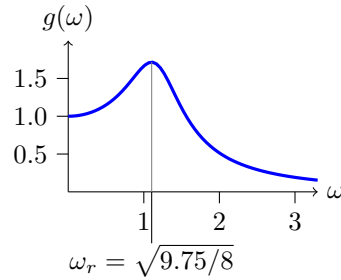
Solution: In Example 9.3 we found the gain function was

$$g(\omega) = \frac{3}{\sqrt{(3 - 2\omega^2)^2 + (1.5\omega)^2}}.$$

To find the practical resonance we have to find the value of ω where $g(\omega)$ has a maximum. There are a few tricks to shorten the algebra, but we'll find the maximum straightforwardly by setting $g'(\omega) = 0$.

$$g'(\omega) = -\frac{3}{2} \cdot \frac{-8\omega(3 - 2\omega^2) + 2(1.5)^2\omega}{((3 - 2\omega^2)^2 + (1.5\omega)^2)^{3/2}} = 0.$$

Setting the numerator to 0 and solving we find $\omega = 0$ or $\omega = \sqrt{9.75/8}$. We require the resonant frequency to be positive, so $\omega_r = \sqrt{9.75/8}$ is the only practical resonant frequency. The graph below shows that this is, in fact, a maximum. (You can also check this using calculus.)



Graph of the gain function with practical resonance marked.

Example 9.6. (A system with no practical resonant frequency.) Consider the system

$$2y'' + 10y' + 3y = 3B \cos(\omega t),$$

where we consider $B \cos(\omega t)$ to be the input. Find all the practical resonant frequencies.

Solution: This is similar to the previous example except that the damping constant is much larger. The algebra will be nearly identical, so we will skip past most of it. The gain is

$$g(\omega) = \frac{3}{\sqrt{(3 - 2\omega^2)^2 + (10\omega)^2}}.$$

So,

$$g'(\omega) = -\frac{3}{2} \cdot \frac{-8\omega(3 - 2\omega^2) + 2(10)^2\omega}{((3 - 2\omega^2)^2 + (10\omega)^2)^{3/2}} = 0.$$

Setting the numerator to 0 and solving for ω we find $\omega = 0$ or $\omega = \sqrt{-11}$. Since neither of these is a positive real number we say that there is **no practical resonant frequency**.

Example 9.7. Consider the system

$$my'' + by' + ky = F_0 \cos(\omega t)$$

where we consider $F_0 \cos(\omega t)$ to be the input. Find all the practical resonant frequencies.

Solution: The sinusoidal solution to this equation is

$$y_p = \frac{F_0 \cos(\omega t - \phi(\omega))}{|P(i\omega)|} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \cos(\omega t - \phi(\omega)), \quad \text{where } \phi(\omega) = \text{Arg}(P(i\omega)).$$

Therefore, the gain (output amplitude/input amplitude) is

$$g(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}.$$

Here we consider the system parameters m , b , k to be fixed, while the gain depends on the input parameter ω .

For this example, we'll show you a standard trick for finding the maximum of $g(\omega)$. The expression for $g(\omega)$ is one over a square root. So $g(\omega)$ has a maximum where the expression under the square root has a minimum. That is, we need to find the minima of

$$h(\omega) = \frac{1}{g^2} = (k - m\omega^2)^2 + b^2\omega^2.$$

Setting the derivative equal to 0 and solving for ω we get

$$h'(\omega) = -4m\omega(k - m\omega^2) + 2b^2\omega = 0.$$

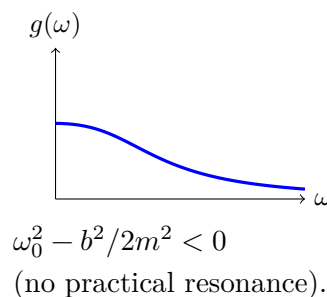
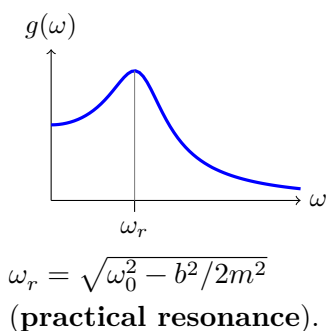
So, $\omega = 0$ or $\omega = \sqrt{k/m - b^2/2m^2}$. Since we require ω_r to be positive we have the following result.

- If $k/m - b^2/2m^2 > 0$ then this system has practical resonance at

$$\omega_r = \sqrt{k/m - b^2/2m^2} = \sqrt{\omega_0^2 - b^2/2m^2}.$$

Here, the last expression gives ω_r in terms of the **natural frequency** $\omega_0 = \sqrt{k/m}$.

- If $k/m - b^2/2m^2 < 0$ then the system does not have a practical resonant frequency.



Notice that in this case if the damping gets too large there is no practical resonance.

For this example, see the mathlet

<https://mathlets.org/mathlets/amplitude-and-phase-second-order-iv/>.

In the text by Edwards and Penney, section 2.7 on radio circuits gives another another example and an application of this.

9.4 The undamped forced system

For a spring-mass system without any damping, we have what is called a **pure resonant frequency**. At this frequency, the amplitude of the response keeps growing to infinity. In this case, we say the **gain is infinite**. We show this with a somewhat general example using symbols for the coefficients.

Example 9.8. Solve the DE $my'' + ky = B \cos(\omega t)$.

Solution: We will only find the particular solution. You can supply the homogeneous solution. We start by doing some calculations we will need later.

1. The natural frequency of the system is $\omega_0 = \sqrt{k/m}$.
2. Characteristic polynomial: $P(r) = mr^2 + k$. We will need both $P(i\omega)$ and $P'(i\omega)$ in polar form.

$$P(i\omega) = k - m\omega^2 = |k - m\omega^2|e^{i\phi(\omega)}, \quad \text{where } \phi(\omega) = \begin{cases} 0 & \text{if } k - m\omega^2 > 0, \text{ i.e., } \omega < \omega_0 \\ \pi & \text{if } k - m\omega^2 < 0, \text{ i.e., } \omega > \omega_0 \end{cases}$$

$$P'(i\omega) = 2im\omega = 2m\omega e^{i\pi/2}.$$

Note that $P(i\omega) = 0$ exactly when $\omega = \sqrt{k/m} = \omega_0$.

Now use the sinusoidal response formula to get

$$y_p(t) = \frac{B \cos(\omega t - \phi(\omega))}{|P(i\omega)|} = \begin{cases} \frac{B \cos(\omega t)}{|k - m\omega^2|} & \text{if } \omega < \omega_0 \\ \frac{B \cos(\omega t - \pi)}{|k - m\omega^2|} & \text{if } \omega > \omega_0 \end{cases}$$

$$y_p(t) = \frac{Bt \cos(\omega_0 t - \phi(\omega_0))}{|P'(i\omega_0)|} = \frac{Bt \cos(\omega_0 t - \pi/2)}{2m\omega_0} \quad \text{if } \omega = \omega_0.$$

Note: In the case $\omega = \omega_0$, we had to use the extended SRF since $P(i\omega_0) = 0$.

Also note, the factor of t in the case $\omega = \omega_0$.

9.4.1 Resonance and amplitude response of the undamped harmonic oscillator

Now let's take $B \cos(\omega t)$ to be the input to the system in the previous example. So the gain (output amplitude/input amplitude) for the system is

$$g(\omega) = \frac{1}{m|\omega_0^2 - \omega^2|}.$$

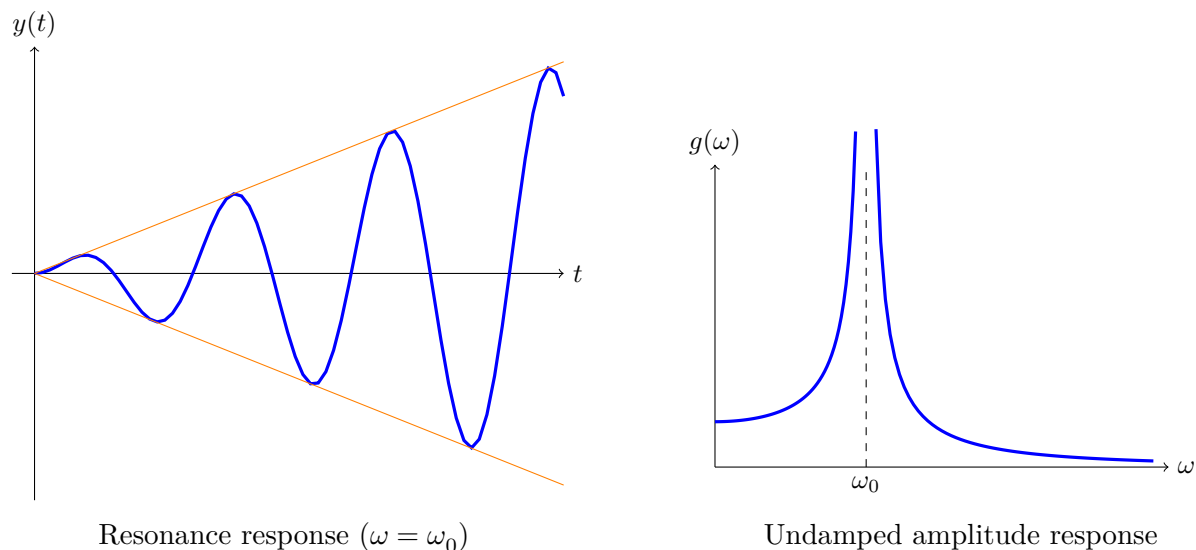
The right hand plot below shows $g(\omega)$ as a function of ω . There is a vertical asymptote at $\omega = \omega_0$. Note that the graph is similar to the graph of the gain for the damped harmonic oscillator except that the peak is infinitely high. Since we don't have a sinusoidal solution when $\omega = \omega_0$ there is no well defined gain at ω_0 . However, given the graphs of the gain and the solution when $\omega = \omega_0$, it is conventional to say that the system has **infinite gain at the frequency $\omega = \omega_0$** .

Let's examine what this means. When $\omega = \omega_0$ we have

$$y_p(t) = \frac{Bt \cos(\omega_0 t - \pi/2)}{2m\omega_0} = \frac{Bt \sin(\omega_0 t)}{2m\omega_0}.$$

This is called **pure resonance**. The natural frequency ω_0 is called the **pure resonant frequency** or simply the **resonant** frequency of the system.

The graph of $y_p(t)$ is shown in the left-hand plot below. Notice that the response is oscillatory but not periodic. The amplitude keeps growing in time because of the factor of t in $y_p(t)$.



Note carefully the different units and different meanings in the plots. The left-hand plot is **output vs. time** for a fixed input frequency. The right-hand plot is **gain vs. input frequency**. $x(t)$ and $g(\omega)$ are in physical units dependent on the system, t is in time, ω is in radians/time.

Physically, for the undamped oscillator, resonance happens because the input force is in sync with the natural frequency of the system and every push adds energy, so the energy in the system keeps growing to infinity. If the input frequency is different from ω_0 , then sometimes the input force acts to add energy and sometimes it removes energy from the system, so the energy stays bounded. Likewise, if there is damping then the damping force is always removing energy from the system and a sinusoidal input can't cause the energy to grow without bound.

9.5 Slight variation of the undamped oscillator

Example 9.9. Consider the system $my'' + ky = f'(t)$ where we take $f(t)$ to be the input and $y(t)$ the response. Solve the DE when $f(t) = B \cos(\omega t)$ and give the gain of the system.

Solution: To find a particular solution we will complexify first and then take the derivative of $f(t)$. This is generally slightly easier than taking the derivative and then complexifying. The complexified DE becomes

$$mz'' + kz = (Be^{i\omega t})' = iB\omega e^{i\omega t}, \quad \text{with } y = \text{Re}(z).$$

As in Example 9.8, we have the following.

The natural frequency of the system is $\omega_0 = \sqrt{k/m}$.

$$P(i\omega) = k - m\omega^2 = |k - m\omega^2|e^{i\phi(\omega)}, \quad \text{where } \phi(\omega) = \begin{cases} 0 & \text{if } k - m\omega^2 > 0, \text{ i.e., } \omega < \omega_0 \\ \pi & \text{if } k - m\omega^2 < 0, \text{ i.e., } \omega > \omega_0 \end{cases}$$

$$P'(i\omega) = 2im\omega = 2m\omega e^{i\pi/2}.$$

Now use the exponential response formula (and its extended version) to get

$$z_p(t) = \begin{cases} \frac{B\omega e^{i\omega t}}{k - m\omega^2} = \frac{B\omega e^{i\pi/2} e^{i\omega t}}{|k - m\omega^2| e^{i\phi(\omega)}} & \text{if } \omega \neq \omega_0, \text{ where } \phi(\omega) = \text{Arg}(k - m\omega^2) \\ \frac{B t i \omega e^{i\omega_0 t}}{2im\omega_0} = \frac{B t \omega e^{i\pi/2} \omega e^{i\omega_0 t}}{2m\omega_0 e^{i\pi/2}} & \text{if } \omega = \omega_0 \end{cases}$$

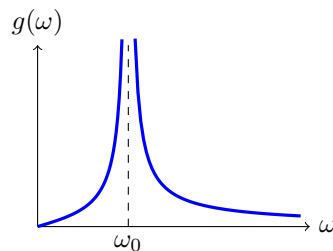
So,

$$y_p(t) = \text{Re}(z_p) = \begin{cases} \frac{B\omega \cos(\omega t + \pi/2)}{|k - m\omega^2|} & \text{if } \omega < \omega_0 \\ \frac{B\omega \cos(\omega t - \pi/2)}{|k - m\omega^2|} & \text{if } \omega > \omega_0 \\ \frac{B t \omega \cos(\omega_0 t)}{2m\omega_0} & \text{if } \omega = \omega_0 \end{cases}$$

Since the input is $B \cos(\omega t)$, we have the gain is

$$g(\omega) = \frac{\omega}{|k - m\omega^2|}$$

As in Example 9.8, there is a vertical asymptote at $\omega = \omega_0$. We also see the gain is 0 when $\omega = 0$. The amplitude response curve is shown below.



9.5.1 Zero-pole diagrams and gain

If there is time we will discuss this in class.

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ES.1803 Differential Equations

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