ES.1803 Problem Section Topics 10-12, Spring 2024
Solutions

1 First-order nonlinear

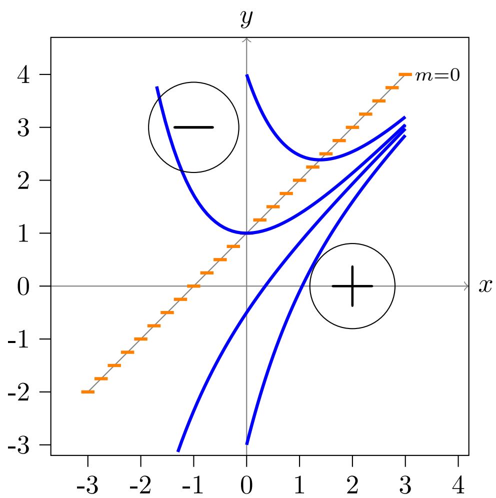
Topic 10: Direction fields, integral curves, existence of solutions

Problem 10.1. *Consider $y' = x - y + 1$.*

(a) *Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.*

Solution: The nullcline is the isocline with $m = 0$. In our case, this is $0 = x - y + 1$, which we also write as $y = x + 1$. Above this line, we have $y > x + 1$ or $0 > x - y + 1$, which means the slope is negative. Below this line, we have $y < x + 1$ and $0 < x - y + 1$ so slope is positive.

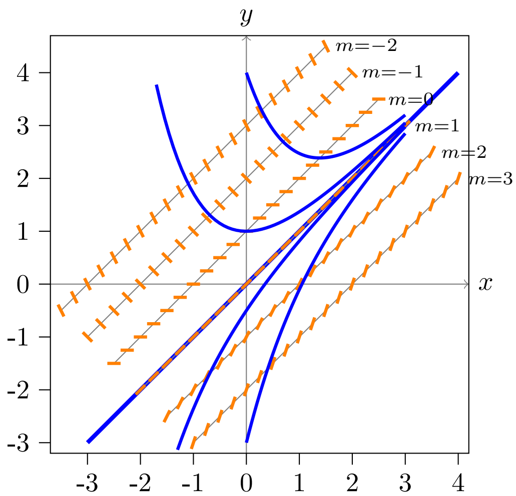
We draw horizontal lines along the nullcline to indicate the slope. The slope field is negative above the line, so integral curves in this region go down towards the nullcline, level off to slope 0 at the nullcline and then turn upwards. The slope field is positive below the nullcline, so integral curves in this region all slope upwards.



(b) *Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.*

(Note the isocline $y = x$ happens to be a solution—don't expect this to happen usually.)

Solution: See graph:



(c) *Can you make a squeezing argument that shows that all solutions go asymptotically to the line $y = x$.*

Solution: This is a little tricky since we will use an indirect argument. We'll consider integral curves below the integral curve $y = x$. The argument for those above $y = x$ is similar.

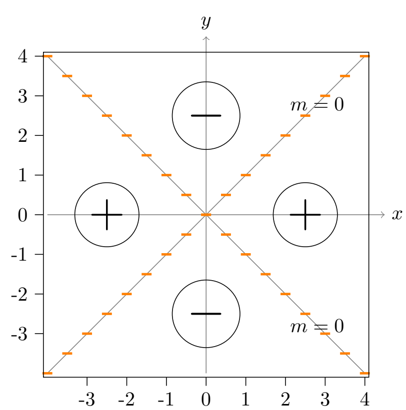
The isoclines are all parallel to the integral curve $y = x$. That is, as lines they have slope 1. The isoclines below the line $y = x$ all have slope field elements of slope greater than 1. The slope of an integral curve below $y = x$ must go asymptotically to 1. (If it stayed greater than $1 + b$, for some positive b, then it would have to keep growing faster than the line $y = x$ and, therefore, cross $y = x$.) If the slope of an integral curve goes asymptotically to 1, the curve must approach the isocline with $m = 1$, i.e. $y = x$.

Problem 10.2. *Consider $y' = x^2 - y^2$*

(a) *Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.*

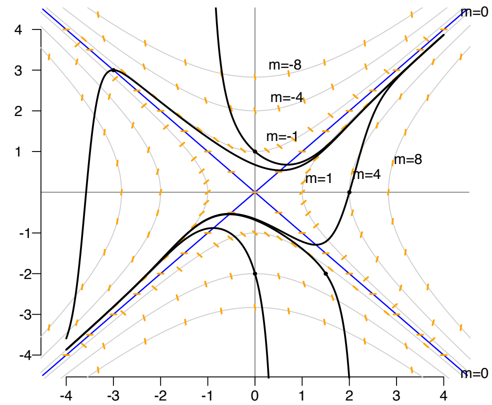
Note: the nullcline consists of two lines.

Solution: The nullcline consists of the lines $y = \pm x$. Below is a sketch of the nullcline with the regions marked + or −. Look at the figure with Part (b) for some integral curves.



(b) *Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.*

Solution: Isoclines are hyperbolas with asymptotes $y = \pm x$.



(c) *Add some integral curves to the plot in Part (b). Include the one with $y(2) = 0$.*

Solution: See plot in Part (b).

(d) *Use squeezing to estimate $y(100)$ for the solution with IC $y(2) = 0$.*

Solution: We can see from the plot in Part (b) that this solution seems to go asymptotically to the nullcline $y = x$.

The argument to see this is a little subtle. We'll give the argument as a sequence of observations. On an exam, you could just state this as an empirical observation about the isoclines sketch.

1. Clearly the nullcline $y = x$ is an upper fence for this integral curve, so the curve stays below this line.

2. To be specific, let's take $m = 2$. The isocline for $m = 2$ goes asymptotically to the line $y = x$. That is, its slope as a curve (not the isocline slope) is close to 1 for large x. Thus, when x is large, the isocline $m = 2$ is a lower fence, i.e., its slope element goes from below to above the isocline.

3. Let x be large enough that the isocline for $m = 2$ is a lower fence. If the integral curve $y(x)$ is below the isocline then its slope is bigger than 2. This means it is growing faster than the isocline and must eventually cross it. At this point it is above the fence and in the funnel between $y = x$ and the isocline for $m = 2$.

4. This funnel goes asymptotically to $y = x$, so we can estimate $y(100) \approx 100$.

Problem 10.3. *Consider $y' = y(1 - y)$ (Note that there is no x; what does this mean for the shape of your nullclines? Your isoclines?)*

(a) *Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as $+$ and negative slope as $-$. Use this to give a very rough sketch of some solution curves.*

Solution: Note. This is secretly introducing autonomous equations.

The sketch is shown with the solution to Part (b).

The nullcline is $0 = y(1-y)$. For this equation to work, we need either $y = 0$ or $y = 1$. Therefore, these two lines are our nullclines. Since the tangent elements lie along the lines, the nullclines turn out to be solutions.

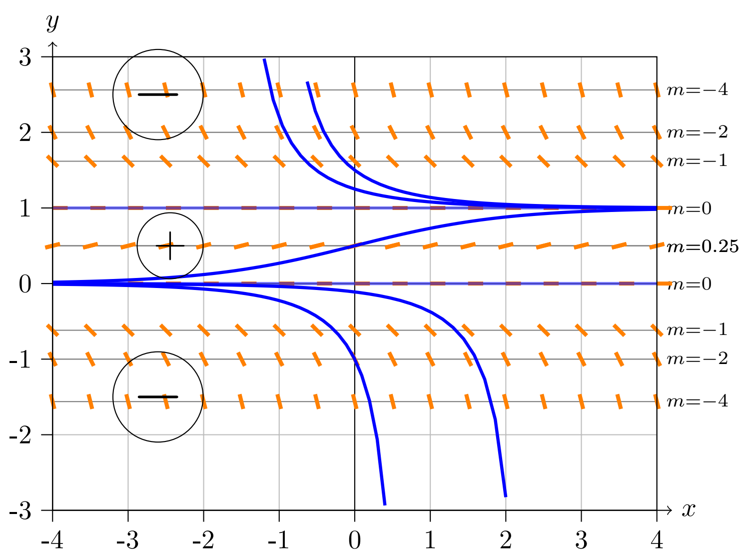
We can see that for $y > 1$, $y' = y(1-y) < 0$, i.e. the slope field is negative. Likewise, for $0 < y < 1$, $y > 0$, so the slope field is positive. Finally, for $y < 0$, $y' < 0$, so the slope field is negative.

Since the nullclines are solutions, no other solutions cannot cross them. This means each solution curve is restricted to one section of the graph. So we have integral curves that come down from $y = \infty$ and go asymptotically to the top nullcline. Likewise, we have integral curves coming up from $y = -\infty$ and going asymptotically to the bottom nullcline. Finally, in the middle section, we have integral curves that come asymptotically from the bottom nullcline and go asymptotically up to the top one. These have a flat S-shape with positive slope. All solutions repeat identically when translated in the horizontal direction.

See the graph in Part (b). Notice that we could come close to drawing it knowing just the nullclines.

(b) *Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.*

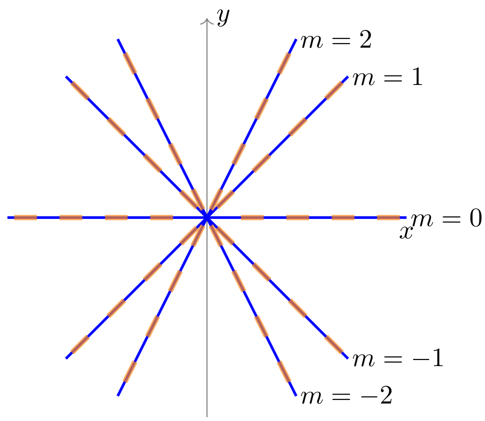
We can see that all isoclines are horizontal lines because the equation for y' does not depend on x and so is constant when y is fixed and x changes.



Problem 10.4. *Consider $y' = y/x$ Note: the line $x = 0$ ($m = \infty$) also separates regions of positive and negative slope.*

(a) *Sketch the isoclines for $m = 0, \pm 1, \pm 2$. Use it this to give a sketch of some solutions.*

Solution: The isoclines are $y/x = m$ or $y = mx$. These are lines that happen to have the same slope as the slope field elements along them. This shows that each of these isoclines is actually a solution.



(b) *This is a rare case where we can solve the DE. Solve the DE and use your solution to draw some integral curves.*

Solution: Separating variables: $\frac{dy}{y} = \frac{dx}{x}$.

Integrating: $\ln|y| = \ln|x| + C$.

Solving for y: $y = Cx$. All solutions are lines through the origin.

Picture is the same as in Part (a).

Note: Because there are no solutions that go through points on the y-axis (other than $(0,0)$), existence of solutions through every point fails. Also, because there are many solutions that go through the origin, uniqueness fails. This is not surpising since $f(x,y) = y/x$ is not continuous when $x = 0$.

Problem 10.5. *For $y' = -y/(x^2+y^2)$, sketch the direction field in the upper half-plane. For the solution with initial condition $y(0) = 1$ explain why you know it is decreasing for $x > 0$. Explain why it is always positive for $x > 0$.*

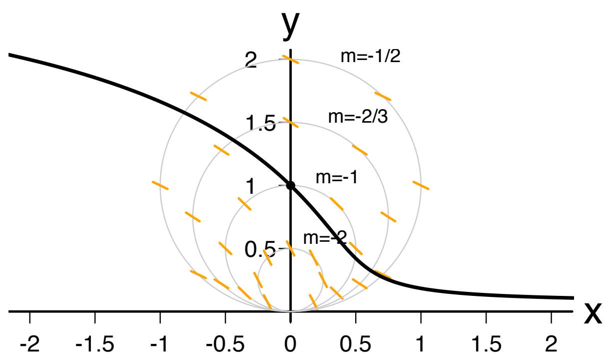
Solution: The isoclines are $-y/(x^2+y^2) = m$. A little algebra converts this to the form

$$x^2 + \left(y + \frac{1}{2m}\right)^2 = \frac{1}{4m^2}.$$

This is a circle of radius $1/(2m)$ centered at the point $(0, -1/(2m))$ on the y-axis. Note that these all go through the origin. This is okay since $-y/(x^2+y^2)$ is not defined at the origin, so all bets are off there.

The isoclines with negative slope ($m < 0$) are in the upper-half plane. We know that because $y' = -y/(x^2+y^2)$ is negative when $y > 0$.

The solution will always be positive because we know that $y(x) = 0$ is a solution (just plug it into the DE). Except at the origin, the existence and uniqueness theorem guarantees that integral curves don't cross. This means an integral curve that starts positive can't cross $y = 0$, so it must stay positive.

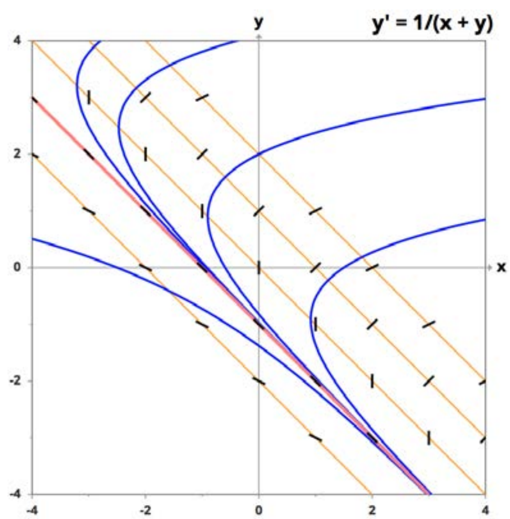


Problem 10.6. *Consider the DE* $y' = \dfrac{1}{x+y}$

Draw a direction field by using about five isoclines; the picture should be square, using the intervals between -4 *and* 4 *on both axes.*

Sketch in the integral curves that pass respectively through (0,0), (-1,1), (0,-2). Will these curves cross the line $y = -x - 1$*? Explain by using the existence and uniqueness theorem*

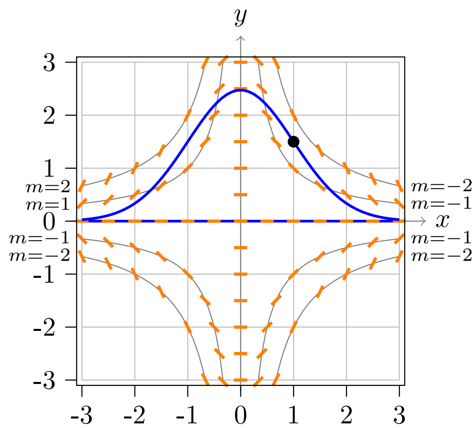
Solution: The isocline for slope m is $\dfrac{1}{x+y} = m$. For $m \neq 0$ this is equivalent to $x + y = 1/m$. These are lines of slope -1. Several are shown in the figure below. The isocline with $m = -1$ is also an integral curve (its slope field elements are all along the line). Since $f(x, y) = 1/(x + y)$ is continuous along the line $x + y = -1$, the existence and uniqueness theorem guarantees that other integral curves can't cross it.



Problem 10.7. *Consider the DE* $y' = -xy$.

(a) *Draw a direction field using isoclines for* $m = 0,\ 1,\ 2,\ -1,\ -2$.

Solution: The nullcline consists of both axes. The isoclines are hyperbolas with two branches, asymptotic to the axes.



(b) *Let $y(x)$ be the solution with initial condition $y(1) = 1.5$. Use fences and funnels to estimate $y(100)$.*

Solution: The x-axis is an integral curve for the solution $y(x) = 0$ This acts as a fence. The isocline for $m = -2$ is an upper fence when $x > 1$. (This is because the slope elements go from above to below the isocline.) Together these two fences form a funnel that goes asymptotically to $y = 0$.

Since the initial point $(1, 1.5)$ is inside the funnel we can estimate $y(100) \approx 0$. The exact value will be slightly bigger.

Topic 11: Numerical methods

Problem 11.8. *For $y' = y^2 - x^2$:*

(a) *Use Euler's method with $h = 0.5$ to estimate $y(3)$ for the solution with initial condition $y(2) = 0$.*

Solution: As in the Topic 11 notes, set up a table with columns: n, x_n, y_n, m, mh.

n	x_n	y_n	m	mh
0	2	0	-4	-2
1	2.5	-2	-2.25	-1.125
2	3	-3.125		

(b) *Is the estimate in Part (a) too high or too low?*

Solution: We can take the derivative of our equation to get the equation for the second derivative $y'' = 2yy' - 2x$. If we look at the point $(x, y) = (2, 0)$, then we can use our original equation to get $y' = -4$, and the second derivative equation to get $y'' = -4 < 0$. A negative second derivative implies the integral curve is concave down, which implies that our estimate is an overestimate, since drawing tangent lines to the curve produces values above the curve.

Problem 11.9. *For $\dfrac{dy}{dx} = F(x, y) = y^2 - x^2$.*

(a) *Use Euler's method to estimate the value at $x = 1.5$ of the solution for which $y(0) = -1$.*

Use step size $h = 0.5$. As in the notes, make a table with columns n, x_n, y_n, m, mh.

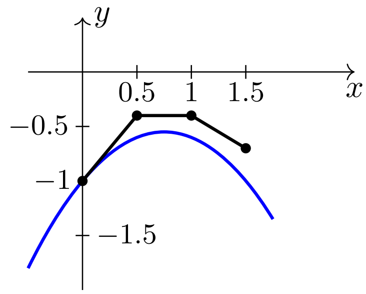
We are estimating y(1.5) using Euler's method with step size 0.5. This takes $\frac{1.5-0}{0.5} = 3$ steps, as outlined in the following table.

n	x_n	y_n	m_n	$m_n h$
0	0	-1	1	0.5
1	0.5	-0.5	0	0
2	1.0	-0.5	-0.75	-0.375
3	1.5	-0.875		

Thus Euler's method gives the estimate

$$y(1.5) \approx y_3 = -0.875.$$

The corresponding Euler polygon for this estimation is



Euler polygon and actual integral curve.

(b) *Is the estimate found in Part (a) likely to be too large or too small?*

It is likely to be too large. One way to see this is to use the second derivative test to check the concavity of solutions around the inital point. First, find $\frac{d^2y}{dx^2} = y''$ by taking the derivative of the differential equation:

$$y'' = \frac{d}{dx}(F(x, y)) = \frac{d}{dx}(y^2 - x^2) = 2yy' - 2x,$$

Second, evaluate the second derivative at the initial point $(0, -1)$ to get

$$y''|_{(0,-1)} = 2(-1)(1) - 2(0) = -2 < 0.$$

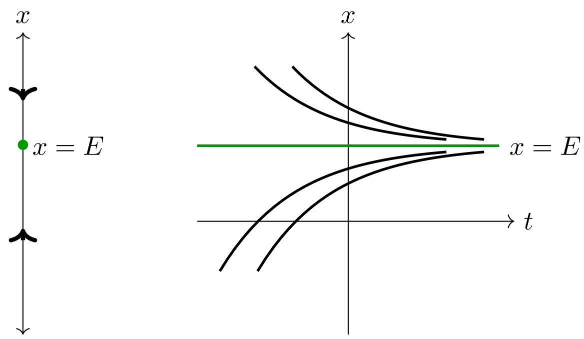
This means the solution that goes through the initial point is concave down. The tangent to a concave down function lies above the function in a small neighborhood, so the Euler estimate for one step is likely to overshoot. Running the same check for the next two endpoints, shows that the second derivative is negative at each endpoint of the Euler polygon. So each of the three steps is likely to overshoot. This suggests the estimate found is likely to be greater than the value of the true solution when $x = 1.5$.

Topic 12: Automonous DEs and bifurcation diagrams

Problem 12.10. *For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.*

Temperature: $x' = -k(x - E)$ (E constant ambient temperature).

Solution: Critical points are when $f(x) = 0$, which in this case is just $x = E$. To draw a phase line, we see that x' is negative for $x > E$ and x' is positive for $x > E$. Looking at the phaseline, we see that $x = E$ is a stable critical point.



If we draw some integral curves, we get curves that head toward the line $x = E$. The intuitive explanation behind this model is that the temperature x heads towards the ambient temperature.

Problem 12.11. *Suppose the following DE models a population $x' = -ax + 1$, which is a constant birth-and-death rate situation modified to include a constant rate of replenishment.*

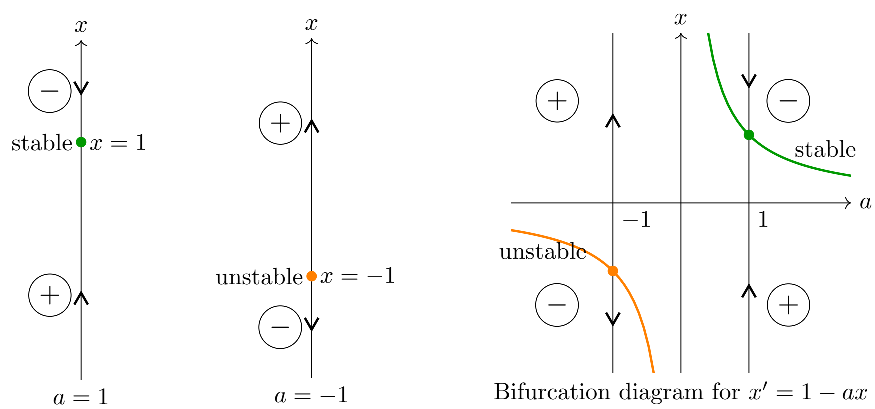
(i) Sketch the bifurcation diagram and list any bifurcation points (these are special values of a).

(ii) The bifurcation points divide the a-axis into intervals. Illustrate one typical case for each interval by giving the phase line diagram. For each of these phase lines, give (rough) sketches of solutions in the tx-plane.

(iii) For what values of a is the population sustainable. What happens for other values of a.

Note the applet 'phase lines' can show this system.

Solution: We answer (i) and (ii) together. The critical points are $x' = -ax + 1 = 0$. So, $x = 1/a$. We graph this in the ax-plane –it's a hyperbola with two branches. Here is the finished bifurcation diagram with two phase lines. These are explained below.



After plotting the critical points, we see that the graph divides the ax-plane into 3 regions. In order to determine the sign of x' in each region, we found phase lines for $a = 1$ and $a = -1$. These are shown at the left. Determining the direction of the arrows was straightforward and we leave it to the reader to supply the details.

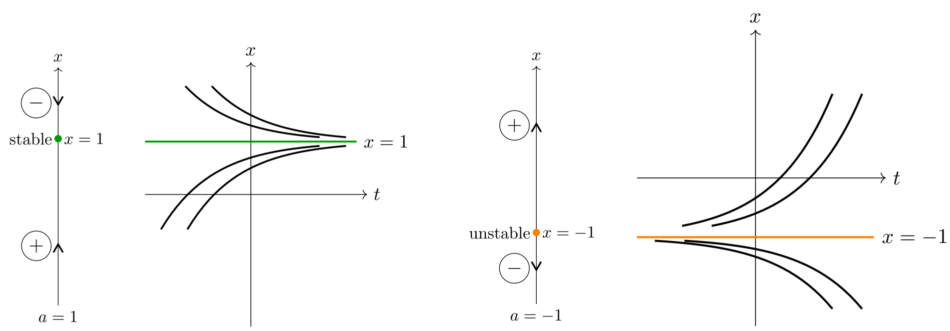
We place the phase lines on the bifurcation diagram at $a = 1$ and $a = -1$. (This answers (ii).) The arrows on the phase lines then tell us the sign of x' in all 3 regions.

Once we know the sign on x', it's a simple matter to decide the stability of each part of the diagram. The stable branch is drawn in green and labeled 'stable'. Likewise, the unstable branch is drawn in orange and labeled 'unstable'.

There is one bifurcation point at $a = 0$. This is a bifurcation point because the bifurcation diagram is qualitatively different on either side of $a = 0$.

(iii) When $a > 0$ there is a positive stable equilibrium, so the population is sustainable. When $a \leq 0$ the population is not sustainable. In fact, it blows up to infinity.

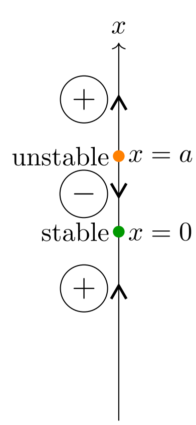
Finally, we do our duty and sketch some solution curves based on the phase lines.



Problem 12.12. *Consider the system $x' = x(x - a) + \frac{1}{4}$, which is the 'doomsday-vs-extinction' equation with the addition of a constant rate of replenishment.*

(a) *First consider the equation $x' = x(x - a)$ with $a > 0$. Why is this called the doomsday-vs-extinction population model?*

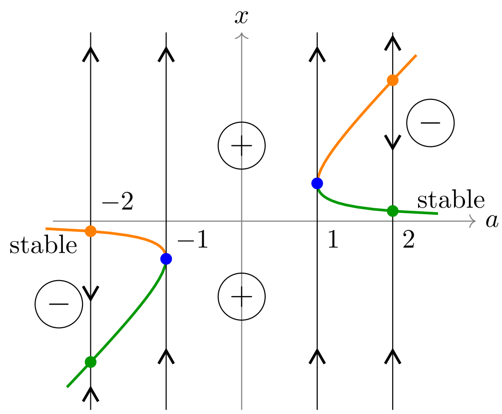
Solution: We draw the phase line. The critical points are $x = 0$ and $x = a$. It is easy to determine the sign of x', these are indicated by the arrows on the phase line.



We see that there is no positive stable equilibrium. If x starts greater than a, then it will increase to infinity –doomsday. If x starts less than a, then it will decrease to 0 –extinction.

(b) *Sketch the bifurcation diagram for $x' = x(x - a) + 1/4$.*

Solution: Here is the bifurcation diagram. It requires a bit of calculus to graph properly and figure the values of various points. This is explained below.



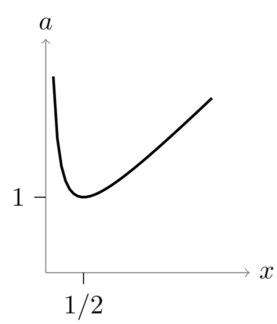
Bifurcation diagram for $x' = x(x - a) + 1/4$

Here is the explanation for the bifurcation diagram.

Computing the crititcal points is simple algebra

$$x' = x(x - a) + \frac{1}{4} = 0 \quad \Rightarrow \quad a = x + \frac{1}{4x}.$$

First notice that this gives a as a function of x. So we'll first plot it with the axes reversed and just for positive x. When x is small a is large. When x is large $a \approx x$. This leads to the graph shown below. We can use calculus to find that the minimum is at $x = 1/2$, $a = 1$. (That is, we solve $\frac{da}{dx} = 0$.)



To get the bifurcation diagram we just interchange the axes. Free of charge, we see that $a = 1$ is the smallest positive value of a on the diagram.

The part of the diagram with $x < 0$ is found similarly.

We then plotted a number of phase lines to identify the regions where x' is positive and negative. As usual, these are marked with + or −. Using these, we can identify the stable and unstable parts of the bifurcation diagram.

(c) *Identify the bifurcation points. For what values of a is the population sustainable? Which positive values of a guarantee against extinction? Which positive values of a guarantee* *against doomsday?*

Solution: The bifurcation points are at $a = \pm 1$.

There are positive stable equilibrium for $a > 1$, so this is the range of a where the population is sustainable.

For $a > 1$, the population either stabilizes at the positive stable equilibrium or blows up to infinity. So, for $a > 1$, x won't go extinct.

For $a = 1$, x will not linger at the semistable equilibrium, instead it is likely to blow up.

For $0 < a < 1$, the population always blows up to infinity. So, for these a, x won't go extinct.

Thus, the population is guaranteed not to go extinct for all $a > 0$.

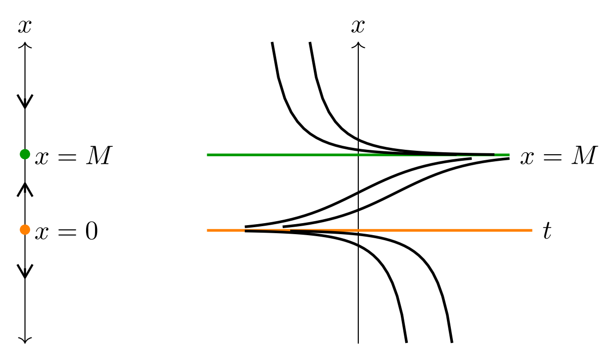
No value of $a > 0$ guarantees against doomsday (x blowing up).

Problem 12.13. *For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.*

Logistic population growth: $x' = kx(M - x)$, *where* $k > 0$

Solution: Critical points are when $f(x) = 0$, which in this case is $x = M$ and $x = 0$. To draw a phase line, we see that

$$x' = kx(M - x) \text{ is } \begin{cases} \text{negative} & \text{for } x > M \\ \text{positive} & \text{for } 0 < x < M \\ \text{negative} & \text{for } x < 0 \end{cases}$$



The phase line shows that $x = M$ is a stable critical point, and $x = 0$ is an unstable critical point.

For the integral curves, we see that above $x = M$ the integral curve has negative slope and goes asymptotically to $x = M$, between $x = 0$ and $x = M$ the curve forms a sort of an s-shape, and for $x < 0$ the curve again has negative slope and curves away from $x = 0$.

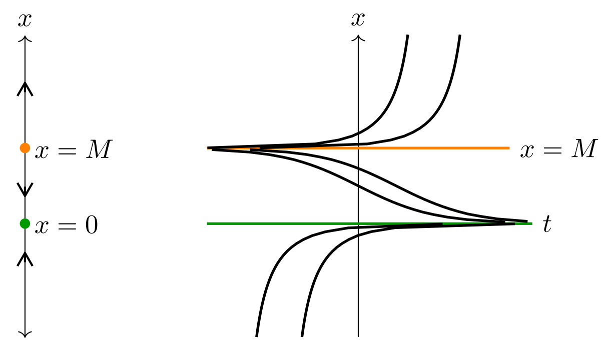
The explanation behind this model is that M is the carrying capacity of the environment: as the population x increases towards M, the growth rate slows, and for populations above $x = M$, animals die off until $x = M$ asymptotically. Populations below zero do not make intuitive sense and we disregard them.

Problem 12.14. *Consider the doomsday-extinction model:* $x' = \beta x^2 - \delta x = kx(x - M)$, *where $\beta, \delta > 0$. Draw the phase line and sketch some integral curves.*

Solution: Critical points: $x' = kx(x - M) = 0 \Rightarrow x = 0,\, M$.

$$x' = kx(x-M) \text{ is } \begin{cases} \text{positive} & \text{for } x > M \\ \text{negative} & \text{for } 0 < x < M \\ \text{positive} & \text{for } x < 0 \end{cases}$$

This gives the following phase line and solution curves.



This means $x = M$ is an unstable critical point, and $x = 0$ is a stable critical point. If we draw some integral curves, we see that above $x = M$ the line has positive slope and away from $x = M$, between $x = 0$ and $x = M$ the line forms a sort of a backwards s-shape with negative slope, and for $x < 0$ the line again has positive slope and curves towards $x = 0$.

The explanation for why this model is of this form is that we assume births are proportional to x^2, i.e., the probability that two randomly roaming members encounter each other and reproduce and we assume that deathrate is constant. We can see that in our solutions, if we start with $0 < x < M$, eventually everything dies, since the birthrate is not enough to overcome the death rate. If we start with $x > M$, then the population soars to infinity because the births are proportional to x^2, which is very large for large x and the constant death rate cannot keep it in check. We disregard the case $x < 0$ because negative numbers of animals don't make sense.