ES.1803 Problem Section Topics 10-12, Spring 2024

1 First-order nonlinear

Topic 10: Direction fields, integral curves, existence of solutions

Problem 10.1. Consider $y' = x - y + 1$.

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as $+$ and negative slope as $-$. Use this to give a very rough sketch of some solution curves.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

(Note the isocline $y = x$ happens to be a solution—don't expect this to happen usually.)

(c) Can you make a squeezing argument that shows that all solutions go asymptotically to the line $y = x$.

Problem 10.2. Consider $y' = x^2 - y^2$

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as $+$ and negative slope as $-$. Use this to give a very rough sketch of some solution curves.

Note: the nullcline consists of two lines.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

(c) Add some integral curves to the plot in Part (b). Include the one with $y(2) = 0$.

(d) Use squeezing to estimate $y(100)$ for the solution with IC $y(2) = 0$.

Problem 10.3. Consider $y' = y(1 - y)$ (Note that there is no x; what does this mean for the shape of your nullclines? Your isoclines?)

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as $+$ and negative slope as $-$. Use this to give a very rough sketch of some solution curves.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

Problem 10.4. Consider $y' = y/x$ Note: the line $x = 0$ ($m = \infty$) also separates regions of positive and negative slope.

(a) Sketch the isoclines for $m = 0, \pm 1, \pm 2$. Use it this to give a sketch of some solutions.

(b) This is a rare case where we can solve the DE. Solve the DE and use your solution to draw some integral curves.

Problem 10.5. For $y' = -y/(x^2 + y^2)$, sketch the direction field in the upper half-plane. For the solution with initial condition $y(0) = 1$ explain why you know it is decreasing for $x > 0$. Explain why it is always positive for $x > 0$.

Problem 10.6. Consider the DE $y' = \dfrac{1}{x+y}$

Draw a direction field by using about five isoclines; the picture should be square, using the intervals between -4 and 4 on both axes.

Sketch in the integral curves that pass respectively through (0,0), (-1,1), (0,-2). Will these curves cross the line $y = -x-1$? Explain by using the existence and uniqueness theorem

Problem 10.7. Consider the DE $y' = -xy$.

(a) Draw a direction field using isoclines for $m = 0, 1, 2, -1, -2$.

(b) Let $y(x)$ be the solution with initial condition $y(1) = 1.5$. Use fences and funnels to estimate $y(100)$.

Topic 11: Numerical methods

Problem 11.8. For $y' = y^2 - x^2$:

(a) Use Euler's method with $h = 0.5$ to estimate $y(3)$ for the solution with initial condition $y(2) = 0$.

(b) Is the estimate in Part (a) too high or too low?

Problem 11.9. For $\dfrac{dy}{dx} = F(x, y) = y^2 - x^2$.

(a) Use Euler's method to estimate the value at $x = 1.5$ of the solution for which $y(0) = -1$.

Use step size $h = 0.5$. As in the notes, make a table with columns n, x_n, y_n, m, mh.

(b) Is the estimate found in Part (a) likely to be too large or too small?

Topic 12: Automonous DEs and bifurcation diagrams

Problem 12.10. For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.

Temperature: $x' = -k(x - E)$ (E constant ambient temperature).

Problem 12.11. Suppose the following DE models a population $x' = -ax + 1$, which is a constant birth-and-death rate situation modified to include a constant rate of replenishment.

(i) Sketch the bifurcation diagram and list any bifurcation points (these are special values of a).

(ii) The bifurcation points divide the a-axis into intervals. Illustrate one typical case for each interval by giving the phase line diagram. For each of these phase lines, give (rough) sketches of solutions in the tx-plane.

(iii) For what values of a is the population sustainable. What happens for other values of a.

Note the applet 'phase lines' can show this system.

Problem 12.12. Consider the system $x' = x(x-a) + \frac{1}{4}$, which is the 'doomsday-vs-extinction' equation with the addition of a constant rate of replenishment.

(a) First consider the equation $x' = x(x-a)$ with $a > 0$. Why is this called the doomsday-vs-extinction population model?

(b) Sketch the bifurcation diagram for $x' = x(x-a) + 1/4$.

(c) Identify the bifurcation points. For what values of a is the population sustainable? Which positive values of a guarantee against extinction? Which positive values of a guarantee against doomsday?

Problem 12.13. For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.

Logistic population growth: $x' = kx(M-x)$, where $k > 0$

Problem 12.14. Consider the doomsday-extinction model: $x' = \beta x^2 - \delta x = kx(x-M)$, where $\beta, \delta > 0$. Draw the phase line and sketch some integral curves.
