Solutions Topics 24-26

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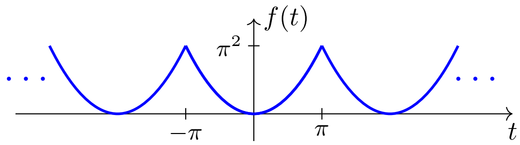
Note: There is a useful integral table on the last page.

Problem 1. (Topic 24) *$f(t)$ has period 2π and $f(t) = t^2$ for $-\pi \le t \le \pi$.*

Solve $2x'' + x' + 18x = f(t)$. (Just find a particular solution.)

Solution: Note, the system has natural frequency $\omega_0 = 3$ and is lightly damped. So we expect near resonance at frequencies near $\omega = 3$.



First, we find the Fourier series of $f(t)$. We have $L = \pi$, $f(t)$ is even, so $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^\pi t^2\cos(nt)\,dt = \frac{4(-1)^n}{n^2} \quad \text{(found with the integral table)}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi t^2\,dt = \frac{2\pi^2}{3}.$$

So, $f(t) = \dfrac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \ldots = \dfrac{\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nt)$.

We have to solve

$$2x'' + x' + 18x = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \ldots$$

Solve in pieces:

$$2x_n'' + x_n' + 18x_n = \cos(nt).$$

Use the SRF: $P(in) = 18 - 2n^2 + in$. So,

$$|P(in)| = \sqrt{(18-2n^2)^2 + n^2}, \quad \boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1}\left(\frac{n}{18-2n^2}\right) \text{ in Q1, Q2}}.$$

Thus, $x_{n,p} = \dfrac{\cos(nt - \phi(n))}{|P(in)|} = \dfrac{\cos(nt-\phi(n))}{\sqrt{(18-2n^2)^2+n^2}}$.

Do $n = 0$ separately: $2x_0'' + x_0' + 18x_0 = 1 \quad \Rightarrow x_0(t) = \dfrac{1}{18}$.

By superposition,

$$x_p(t) = \frac{a_0}{2}x_0(t) + \sum_{n=1}^{\infty} a_n x_n(t) = \frac{\pi^2}{54} + \sum_{n=1}^{\infty}\frac{4(-1)^n\cos(nt-\phi(n))}{n^2\sqrt{(18-2n^2)^2+n^2}}.$$

Problem 2. (Topic 25)

(a) *Find the general solution to the PDE with BC:*

PDE: $y_{tt} = c^2 y_{xx}, \quad 0 \le x \le L, \quad t \ge 0$
BC: $y_x(0,t) = 0, \quad y_x(L,t) = 0$

Solution: Guess a separated solution: $u(x,t) = X(x)T(t)$.

Plug into PDE: $XT'' = c^2 X'' T \quad \Rightarrow \dfrac{T''(t)}{c^2 T(t)} = \dfrac{X''(x)}{X(x)} = \text{ constant } = -\lambda.$

This gives two ODEs: $X'' + \lambda X = 0, \quad T'' + c^2 \lambda T = 0.$

Note: For separated solutions, the BC are $X'(0) = 0, \quad X'(L) = 0.$

Case $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x), \quad T(t) = d\cos(c\sqrt{\lambda}\,t) + e\cos(c\sqrt{\lambda}\,t).$

So, $X'(x) = -\sqrt{\lambda}\,a\sin(\sqrt{\lambda}\,x) + \sqrt{\lambda}\,b\cos(\sqrt{\lambda}\,x).$

Boundary conditions:

BC: $X'(0) = \sqrt{\lambda}\,b = 0 \;\Rightarrow\; b = 0.$ So, $X'(x) = -\sqrt{\lambda}\,a\sin(\sqrt{\lambda}\,x).$

BC: $X'(L) = -\sqrt{\lambda}\,a\sin(\sqrt{\lambda}\,L) = 0.$

The nontrivial solutions need $\sin(\sqrt{\lambda}\,L) = 0 \quad \Rightarrow \sqrt{\lambda} = \dfrac{n\pi}{L}$, $n = 1, 2, 3, \ldots$.

For $\sqrt{\lambda} = \dfrac{n\pi}{L}$, we write $X_n(x) = a_n \cos\left(\dfrac{n\pi}{L}x\right)$ and $T_n = d_n \cos\left(\dfrac{cn\pi}{L}t\right) + e_n \sin\left(\dfrac{cn\pi}{L}t\right)$

Thus the modal solutions are

$$u_n(x,t) = X_n(x)T_n(t) = \cos\left(\frac{n\pi}{L}x\right)\left[d_n \cos\left(\frac{cn\pi}{L}t\right) + e_n \sin\left(\frac{cn\pi}{L}t\right)\right].$$

(We drop a_n because it's redundant.)

Case $\lambda = 0$: $X(x) = a + bx, \quad T(t) = d + et.$

Boundary conditions:

$$\left.\begin{aligned} X'(0) &= b = 0 \\ X'(L) &= b = 0 \end{aligned}\right\} \quad \Rightarrow X_0(x) = a_0,\; T_0(t) = d_0 + e_0 t.$$

So we have one more modal solution: $u_0(x,t) = X_0(x)T_0(t) = \frac{d_0}{2} + \frac{e_0}{2}t$. (Again we drop a_0 because it's redundant. We wrote the constants as $\frac{d_0}{2}$ and $\frac{e_0}{2}$ because of the cosine series coming in Part (b).)

By superposition, the general solution to the PDF and BC is

$$\boxed{u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) = \frac{d_0}{2} + \frac{e_0 t}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\left[d_n \cos\left(\frac{cn\pi\,x}{L}\right) + e_n \cos\left(\frac{cn\pi\,x}{L}\right)\right]}.$$

(b) *Find the solution satisfying the initial conditions* $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Solution: Plug in $t = 0$ to the solution in Part (a). The first initial condition is

$$u(x,0) = \underbrace{\frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi x}{L}\right)}_{\text{cosine series for } f(x)} = f(x).$$

So,

$$\boxed{d_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \quad n = 1,\,2,\,3,\,...}$$

$$\boxed{d_0 = \frac{2}{L}\int_0^L f(x)\,dx}$$

The second initial condition is

$$u_t(x,0) = \underbrace{\frac{e_0}{2} + \sum_{n=1}^{\infty} \left(e_n \cdot \frac{cn\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)}_{\text{cosine series for } g(x)}$$

The entire circled expression is the Fourier cosine coefficient. So,

$$e_n \cdot \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \quad \Rightarrow \boxed{e_n = \frac{2}{cn\pi}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \quad n = 1,\,2,\,3,\,...}$$

$$\boxed{e_0 = \frac{2}{L}\int_0^L g(x)\,dx}$$

All together, the boxed expressions specify the full solution to the PDE with BC and IC.

Problem 3. (Topic 25)

Find the general solution to the inhomogeneous PDE with inhomogeneous BC

PDE: $u_t = 4u_{xx} + e^{-x}, \quad 0 \le x \le \pi, \quad t \ge 0$

BC: $u(0,t) = 2, \quad u(\pi,t) = 3$

Solution: Strategy: Find a particular solution and the general homogeneous solution, then use superposition.

Particular solution: The input is steady-state, so guess a steady-state solution, i.e., $u(x,t) = X(x)$.

Plug into PDE: $0 = 4X''(x) + e^{-x} \quad \Rightarrow X''(x) = -\frac{e^{-x}}{4} \quad \Rightarrow X(x) = -\frac{1}{4}e^{-x} + c_1 x + c_2$

Match the BC:

$$X(0) = -1/4 + c_2 = 2 \quad \Rightarrow c_2 = 9/4.$$

$$X(\pi) = -e^{-\pi}/4 + c_1\pi + c_2 = -e^{-\pi}/4 + c_1\pi + 9/4 = 3 \quad \Rightarrow c_1 = \frac{3 + e^{-\pi}}{4\pi}.$$

So, $\boxed{u_p(x,t) = -\frac{e^{-x}}{4} + \left(\frac{3+e^{-\pi}}{4\pi}\right)x + \frac{9}{4}}$.

General homogeneous solution $u_h(x,t)$:

Homogeneous PDE: $(u_h)_x = 4(u_h)_{xx}$

Homogeneous BC: $u_h(0,t) = 0, \quad u_h(\pi,t) = 0$

We have solved this many times:

$$\boxed{u_h(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-4n^2 t}}.$$

Thus the full solution to the inhomogeneous PDE and BC is

$$\boxed{u(x,t) = u_p(x,t) + u_h(x,t) = -\frac{e^{-x}}{4} + \left(\frac{3+e^{-\pi}}{4\pi}\right) x + \sum_{n=1}^{\infty} b_n \sin(nx) e^{-4n^2 t}}.$$

For completeness we show how to find u_h.

Try a separated solution $u_h(x,t) = X(x)T(t)$.

Plug into the PDE: $XT' = 4X''T \quad \Rightarrow \dfrac{T'}{4T} = \dfrac{X''}{X} = \text{ constant } = -\lambda$

This gives two ODEs: $X'' + \lambda X = 0, \quad T' + 4\lambda T = 0.$

<u>Case $\lambda > 0$</u>: Solve the ODE: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$

Consider the BC:

$$\begin{aligned} X(0) &= a = 0 \qquad \Rightarrow X(x) = b\sin(\sqrt{\lambda}x) \\ X(\pi) &= b\sin(\sqrt{\lambda}\pi) = 0 \qquad \Rightarrow \sqrt{\lambda} = n, \quad n = 1,\, 2,\, 3,\, \ldots \end{aligned}$$

Since $T(t) = ce^{-4\lambda t}$, we have modal solutions $\boxed{u_n(x,t) = b_n \sin(nx) e^{-4n^2 t}}$.

<u>Case $\lambda = 0$</u>: Solve the ODE: $X(x) = a + bx$.

BC: $\left.\begin{aligned} X(0) &= a = 0 \\ X(\pi) &= b\pi = 0 \end{aligned}\right\} \quad \Rightarrow a = 0,\, b = 0 \quad \Rightarrow \text{ only trivial solutions}$

<u>Case $\lambda < 0$</u>: Always only trivial solutions.

So the general homogeneous solution is

$$u_h(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-4n^2 t}.$$

(Exactly what we have above.)

Integrals *(for n a positive integer)*

1. $$\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$$

2. $$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$$

3. $$\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$$

4. $$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$$

1′. $$\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$$

2′. $$\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \textit{for } n \textit{ odd} \\ 0 & \textit{for } n \neq 0 \textit{ even} \end{cases}$$

3′. $$\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \textit{for } n \textit{ odd} \\ \dfrac{-\pi^2}{n} & \textit{for } n \neq 0 \textit{ even} \end{cases}$$

4′. $$\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$$

If $a \neq b$

5. $$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

6. $$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

7. $$\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$$

8. $$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$

9. $$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$

10. $$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$