# Complexity of Games & Puzzles

[Demaine, Hearn & many others]

<table>
<thead>
<tr>
<th>Complexity</th>
<th>0 players (simulation)</th>
<th>1 player (puzzle)</th>
<th>2 players (game)</th>
<th>Team, imperfect info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded</td>
<td>(\mathbb{P})</td>
<td>(\mathbb{NP})</td>
<td>(\mathbb{PSPACE})</td>
<td>(\text{NEXPTIME})</td>
</tr>
<tr>
<td>Unbounded</td>
<td>(\mathbb{PSPACE})</td>
<td>(\mathbb{PSPACE})</td>
<td>(\text{EXPTIME})</td>
<td>Undecidable bridge?</td>
</tr>
</tbody>
</table>

- **\(\mathbb{P}\)**: Polynomial time
- **\(\mathbb{NP}\)**: Nondeterministic Polynomial time
- **\(\mathbb{PSPACE}\)**: Polynomial space
- **\(\text{EXPTIME}\)**: Exponential time
- **Undecidable**

*Images courtesy of various contributors.*
Constraint Logic
[Hearn & Demaine 2009]
Constraint Graphs

Machine = graph, red & blue edges
Constraint Graphs

Machine state = orientation

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Constraint Logic

Rule: at least 2 units incoming at a vertex

Move: reverse an edge, preserving Rule

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**AND vertex**

Rule: at least 2 units incoming at a vertex

not your usual AND gate!
SPLIT vertex

Rule: at least 2 units incoming at a vertex
**OR vertex**

**Rule:** at least 2 units incoming at a vertex

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Decision Problem

can you reverse this edge?
# Constraint Logic

[Hearn & Demaine 2009]

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Decision Problem

can you reverse this edge?

Theorem:
PSPACE-complete

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Sliding-Block Puzzles

Courtesy of Dr. Jim Storer. Used with permission.
Sliding-Block Puzzles
[Hearn & Demaine 2002]

Corollary:
PSPACE-complete
Sliding-Block Puzzles
[Hearn & Demaine 2002]

Corollary: PSPACE-complete
Wiring Vertices Together

AND wants red
OR wants blue

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Red-Blue Conversion

assume an even number of conversions
Red-Blue Conversion

assume an even number of conversions
Boolean Formulas

\[(w \lor x \lor y) \land (w \lor \overline{x} \lor z) \land (w \lor \overline{y} \lor z)\]
Quantified Boolean Formulas (QBF)

\[ \forall x \exists y \forall w \cdots \exists z [(x \lor y) \land \cdots \land (\neg z \lor x \lor \neg w)] \]

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Existential Quantifier
Universal Quantifier

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Latch

unlocked

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Universal Quantifier
Crossover Gadget
OR from Protector OR

(b) Protected OR
Rush Hour
[Hearn & Demaine 2002]

PSPACE-completeness known [Flake & Baum 2002]
Triangular Rush Hour

[a] AND vertex
[b] Connector
[c] OR vertex

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Open: 1×1 Rush Hour
[Tromp & Cilibrasi 2008]

- P or PSPACE-complete or ...?
Plank Puzzles [Hearn 2004]

(a) AND

(b) OR

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Sokoban
[Hearn & Demaine 2002]

PSPACE-completeness known [Culberson 1998]
Push-2F
[Demaine, Hearn, Hoffmann 2002]
Constraint Logic
[Hearn & Demaine 2009]

0 players
(simulation)
1 player
(puzzle)
2 players
(game)

team,
imperfect info

bounded

unbounded

PSPACE
PSPACE
EXPTIME
Undecidable

P
NP
PSPACE
NEXPTIME

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Constraint Logic
[Hearn & Demaine 2009]

bounded

PSPACE
bounded

PSPACE

EXPTIME

Undecidable

0 players
(simulation)

P

1 player
(puzzle)

NP

2 players
(game)

PSPACE

team, imperfect info

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Amazons
[Hearn 2005]

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