Problem Set 3

DUE: 3/1/07

Problem 1

Consider the continuous-time function, $\Lambda(t)$, given by

$$\Lambda(t) = \begin{cases} 
1 - |t|/T & \text{if } -T < t < T \\
0 & \text{otherwise}
\end{cases}$$

(a) Show that $\Lambda(t)$ can be thought of as resulting from the convolution of a rectangular window with itself, that is,

$$\Lambda(t) = c \times \Pi_T(t) \ast \Pi_T(t),$$

where

$$\Pi_T(t) = \begin{cases} 
1 & \text{if } -T < t < T \\
0 & \text{otherwise}
\end{cases}$$

Specify the values of $c$ and $T$ in terms of $T_s$.

(b) Determine $\Lambda(F)$, the CTFT of $\Lambda(t)$.

Now consider the continuous-time function, $\phi(t)$, given by

$$\phi(t) = \frac{\sin \pi F_s t}{\pi F_s t}.$$

(c) Sketch $\phi(t)$ and $\Lambda(t)$ on the same coordinates.

(d) Determine $\Phi(F)$, the CTFT of $\phi(t)$, and then sketch $\Phi(F)$ and $\Lambda(F)$ on the same coordinates.

As we saw in Chapter 1, the sampling theorem states that any bandlimited continuous-time signal can, in principle, be exactly reconstructed from its samples by means of the interpolation formula,

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \phi(t - nT_s). \quad (1)$$

In many cases this formula is not practical because it requires a summation over the entire duration of the sampled signal. Here we will consider the effect of replacing the interpolation function $\phi(t)$ by the finite-duration function $\Lambda(t)$.

(e) Consider using $\Lambda(t)$ in place of $\phi(t)$ to reconstruct samples of the signal $x(t) = \sin(2\pi Ft)$. Sketch the original $x(t)$, along with the reconstructed signals for the two conditions corresponding to the sampling rates $T_s = \frac{1}{4F}$ and $T_s = \frac{1}{8F}$. What conditions are required to obtain a reasonably good approximation using $\Lambda(t)$?
(f) Discuss how the two interpolation functions differ in the frequency domain. What frequency domain features make the reconstruction based on \( \Lambda(t) \) less accurate? (Note that the reconstruction described by Eq. (1) can be interpreted as the time-domain convolution of the sampled signal and the interpolation function. Also, you may find it helpful to sketch \( X(F) \), the CTFT of a continuous-time sine wave, \( x(t) \), and \( X(f) \), the DTFT of \( x[n] \) obtained by sampling \( x(t) \).)

**Problem 2**

(a) Find a closed-form expression for \( X(F) \), the CTFT of the amplitude-modulated signal

\[
x(t) = (1 + m \cos 2\pi F_m t) \cos 2\pi F_c t.
\]

*Hint:* Show that \( x(t) \) can be written as

\[
x(t) = \cos 2\pi F_c t + \frac{m}{2} \cos 2\pi (F_c - F_m) t + \frac{m}{2} \cos 2\pi (F_c + F_m) t
\]

(b) Assume that \( F_c, F_m, \) and \( m \) are unknown. You propose to measure these three parameters using the following method:

1. Sample \( x(t) \) at sampling frequency \( F_s = 5000 \) Hz for a duration of \( T = 20 \) ms. This gives the finite, discrete-time signal \( x[n] \).

2. Compute the N=100 point DFT of of \( x[n] \). This gives \( X[k] \) for \( k = 0, \ldots, 99 \).

For each of the following five cases, determine whether there is sufficient information in \( |X[k]| \) to estimate \( F_c, F_m, \) and \( m \) unambiguously. If yes, describe your method for estimating these parameters and give numerical values for your estimates. If not, specify how you would modify the measurement parameters \( T, F_s, \) and \( N \) in order to obtain unambiguous estimates. Feel free to use Matlab or any other software to compute and sketch \( |X[k]| \).

i) \( F_c = 1475 \) Hz, \( F_m = 200 \) Hz, \( m = 1 \)

ii) \( F_c = 1475 \) Hz, \( F_m = 1200 \) Hz, \( m = 1 \)

iii) \( F_c = 1475 \) Hz, \( F_m = 40 \) Hz, \( m = 1 \)

iv) \( F_c = 1475 \) Hz, \( F_m = 200 \) Hz, \( m = 0.05 \)

v) \( F_c = 1500 \) Hz, \( F_m = 200 \) Hz, \( m = 0.05 \)
Problem 3

In this problem we will consider two $N$-point FIR filters, $h_1[n]$ and $h_2[n]$, formed from the $(\frac{N}{2} + 1)$-point sequence $h[n]$ as follows:

$h_1[n] = \begin{cases} 
  h[n], & 0 \leq n \leq \frac{N}{2} \\
  h[N - n], & \frac{N}{2} + 1 \leq n \leq N - 1 
\end{cases}$

and

$h_2[n] = \begin{cases} 
  h[\frac{N}{2} - n], & 0 \leq n \leq \frac{N}{2} \\
  h[n - \frac{N}{2}], & \frac{N}{2} + 1 \leq n \leq N - 1 
\end{cases}$

(a) Sketch $h_1[n]$ and $h_2[n]$ for $N = 8$ and

$$h[n] = \begin{cases} 
  \left(\frac{1}{2}\right)^n, & 0 \leq n \leq 4 \\
  0, & \text{otherwise}
\end{cases}$$

(b) For arbitrary values of $N$, express the relationship between $h_1[n]$ and $h_2[n]$ in terms of cyclic convolution.

(c) Determine the relationship between $H_1[k]$ and $H_2[k]$, the DFTs of $h_1[n]$ and $h_2[n]$.

(d) We want to filter a finite duration signal, $x[n]$, with $h_2[n]$ from (a). We have decided to do this by computing the $M$-point DFT of $h_2[n]$, computing the $M$-point DFT of $x[n]$, multiplying the two DFTs, and computing the $M$-point inverse DFT of the product. Given that $x[n]$ is zero outside the range $0 \leq n \leq L - 1$, and $L = 1000$, what is the minimum value of $M$ that will produce the desired result? What value of $M$ should be used with a standard FFT algorithm that requires $M = 2^n$?