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Chapter 5 - SAMPLING IN TIME AND FREQUENCY

Introduction

In previous chapters, we studied how discrete-time signals can be obtained by sampling continuous-time signals, and then considered the design and analysis of digital filters in both the time and frequency domains. This chapter considers the operation of sampling in more detail. In particular, we will study the operations of both sampling in time and sampling in frequency, and examine the effects of sampling in one domain on the representation of the signal in the other domain. Using the concept of sampling, we will establish the relationships between the different Fourier transforms (CTFT, DTFT, CTFS, DFS, DFT). Finally, we will consider two important applications of sampling; implementing continuous-time LTI systems with digital filters is based on sampling in time, while spectral analysis is based on sampling in frequency.

5.1 Sampling in time

5.1.1 Discrete-time signals as special continuous-time signals

Before discussing sampling in time, it is useful to establish an additional relationship between discrete-time signals and continuous-time signals. In particular, discrete-time signals can be considered as a special case of continuous-time signals, specifically, a weighted sum of impulses spaced at regular intervals. To see this, consider a discrete-time signal $x[n]$ with DTFT $X(f)$, and define the continuous-time signal $x_s(t)$ as a sum of shifted impulses weighted by the values of $x[n]$:

$$x_s(t) \triangleq \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \tag{5.1}$$

The CTFT of this signal is

$$\begin{aligned} X_s(F) &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \right] e^{-j2\pi Ft} dt \\ &= \sum_{n=-\infty}^{\infty} x[n] \left[\int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j2\pi Ft} dt \right] \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi FnTs} = X(f)|_{f=FT_s=\frac{F}{F_s}}. \end{aligned}$$

Thus, when $x_s(t)$ and $x[n]$ are related as in (5.1), the CTFT of $x_s(t)$ is identical to the DTFT of $x[n]$. Because the relation between a signal and its transform is unique, this means that $x[n]$

and $x_s(t)$ can be considered as different representations of the same signal. In other words, one interpretation of discrete-time signals is as the weighted sum of continuous-time impulses, $\delta(t - nT_s)$, spaced at regular intervals equal to the sampling period.

5.1.2 Summary of sampling in time

The operation of sampling a continuous-time signal is equivalent to multiplication with a periodic impulse train, followed by conversion from continuous-time impulses to discrete-time impulses. Figure 1(a) shows an arbitrary continuous-time signal bandlimited to W, $x(t)$, and a representation of its CTFT, $X(F)$. Figure 1(b) shows the sampling function $p(t)$ with sampling period $T_s = \frac{1}{F_s}$, and its CTFT, $P(F)$. The sampling function is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

and its CTFT is (3.20l)

$$P(F) = F_s \sum_{k=-\infty}^{\infty} \delta(F - kF_s).$$

Figure 1(c) shows the sampled signal in continuous time, $x_s(t) = x(t)p(t)$, where the sampled signal is simply the product of the original signal and the sampling function $p(t)$. Sampling a signal in time corresponds to forming a periodic signal in frequency. This can be understood as a result of the product theorem and the above observation, based on Eq. (3.20l), that the transform of a periodic impulse train in time is a periodic impulse train in frequency. From the product theorem, multiplication in the time domain corresponds to convolution in the frequency domain. Therefore, sampling (multiplication by a periodic impulse train) in time corresponds to forming a periodic signal (convolution with a periodic impulse train) in frequency. We can show that the sampled signal in the frequency domain consists of shifted replicas of the original CTFT, scaled by the sampling frequency, as given by

$$\begin{aligned} X_s(F) &= X(F) * P(F) \\ &= \int_{-\infty}^{\infty} X(\phi)P(F-\phi)\, d\phi = F_s \int_{-\infty}^{\infty} X(\phi) \sum_{k=-\infty}^{\infty} \delta(F - \phi - kF_s)\, d\phi \\ &= F_s \sum_{k=-\infty}^{\infty} X(F - kF_s). \end{aligned} \tag{5.2}$$

Figure 1(d) shows the final step in sampling which consists of converting the sampled, continuous-time signal $x_s(t)$, to the discrete-time signal, $x[n]$. This step does not correspond to any physical operation performed on the signal, rather, it is a conceptual step. It is based on the relationship described in Eq. (5.1) and allows substitution of the discrete-time index, n, for the continuous-time index t. The time index $t = 0, T_s, 2T_s, \ldots$ becomes $n = 0, 1, 2, \ldots$ and correspondingly, the frequency axis is scaled by T_s. This converts the CTFT in Fig. 1(c) to the DTFT in Fig. 1(d). The DTFT is periodic with period 1, as expected.

5.1.3 Nyquist's sampling theorem revisited

The Nyquist sampling theorem states that, if a continuous-time signal $x(t)$ is such that its spectrum $X(F)$ is zero for $|F| > W$, then it can be exactly reconstructed from samples taken at a frequency $F_s > 2W$. This reconstruction is accomplished using the interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi F_s(t - nT_s)}{\pi F_s(t - nT_s)} \tag{5.3}$$

where $T_s = \frac{1}{F_s}$. In the Appendix to Chapter 1, a proof of this theorem for periodic signals was presented and then extended to non-periodic signals. In the following discussion, a direct proof for non-periodic signals is given based on our understanding of the sampling operation in both the time and frequency domains.

Equation (5.2) and its representation in Fig. 1(c) show that frequency aliasing is avoided if the sampling rate exceeds twice the highest frequency component in $x(t)$, that is, if $F_s = \frac{1}{T_s} > 2W$. When this condition is satisfied, there is no overlap between the $X(F - kF_s)$, which are frequency-translated versions of $X(F)$, and therefore, over the range $\frac{-F_s}{2} \leq F \leq \frac{F_s}{2}$ we have exactly

$$X_s(F) = F_s X(F) \quad \text{for} \quad |F| \leq \frac{F_s}{2}.$$

As shown in Figure 2, we can reconstitute the original signal by lowpass filtering $x_s(t)$ to keep only frequencies below $\frac{F_s}{2}$ for which the spectra of the two signals $x_s(t)$ and $x(t)$ are the same within a multiplicative constant:

$$X(F) = \frac{1}{F_s} X_s(f)\, \Pi_{\frac{F_s}{2}}(F) \tag{5.4}$$

Using

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s),$$

which comes from Eq. (5.1) together with the definition of sampling, $x[n] = x(nT_s)$, we take the inverse transform of Eq. (5.4). Making use of the convolution theorem and of Eq. (3.20h) for the inverse CTFT of an ideal lowpass filter (rectangle in frequency) yields

$$\begin{aligned} x(t) &= \frac{1}{F_s} x_s(t) * \frac{\sin \pi F_s t}{\pi t} = \left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \right] * \frac{\sin \pi F_s t}{\pi F_s t} \\ &= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi F_s(t - nT_s)}{\pi F_s(t - nT_s)}. \end{aligned} \tag{5.5}$$

But Eq. (5.5) is the same as Eq. (5.3), which completes the proof of the sampling theorem. This reconstruction of the continuous-time signal by lowpass filtering of the sampled signal is illustrated in Fig. 2.

5.2 Sampling in frequency

5.2.1 Summary of sampling in frequency

Like sampling in time, sampling in frequency is equivalent to multiplication with a periodic impulse train, followed by conversion from continuous impulses to discrete impulses. Figure 3(a) shows an $x_1[n]$, an arbitrary discrete-time signal of duration N_0, and $X_1(F)$, a representation of its DTFT. Figure 3(b) shows the frequency sampling function $P(f)$ with sampling period $\frac{1}{N}$, and its representation in the time domain, $p[n]$. $P(f)$ and $p[n]$ are related by the DTFT pair (3.12m)

$$p[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

and

$$P(f) = \frac{1}{N} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{N}).$$

Figure 3(c) shows the signal after sampling in frequency. In the frequency domain, the sampled signal is simply the product $X_1(f)P(f)$. Sampling a signal in frequency corresponds to forming a periodic signal in time. This results from the convolution theorem and the fact that the transform of a periodic impulse train in frequency is a periodic impulse train in time, analogous to the corresponding observation made above for sampling in time. Using the convolution theorem, we can show that the frequency-sampled signal in the time domain is given by

$$x_1[n] * p[n] = \sum_{k=-\infty}^{\infty} x_1[k]p[n-k] = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{r=-\infty}^{\infty} \delta[n-k-rN] \tag{5.6}$$

$$= \sum_{r=-\infty}^{\infty} x_1[n - rN]. \tag{5.7}$$

As expected, the result of sampling in frequency has been to create a signal that is periodic in time. Time aliasing is avoided if the spacing of impulses in time is at least the duration of the time domain signal, that is, $N \geq N_0$. When this condition is satisfied, there is no overlap between adjacent copies of $x_1[n - rN]$, the time-shifted versions of $x_1[n]$.

Figure 3(d) shows the conversion to discrete frequency representation using the discrete frequency index k. Like the similar transformation performed between Figs. 1(c) and (d), this does not correspond to any physical operation on the signal, but is a conceptual step that consists of rescaling the time and frequency axes. Defining the frequency samples $X_1[k]$ to be the values of the frequency-sampled signal at $k = fN$,

$$X_1[k] \triangleq X_1(f)P(f) = X_1(f)\frac{1}{N} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{N}) \tag{5.8}$$

$$= \frac{1}{N} X_1(\frac{k}{N}). \tag{5.9}$$

These relationship between these time and frequency samples is governed by the discrete Fourier series (DFS), which applies to signals that are discrete and periodic in time, and results in a representation that is also discrete and periodic in frequency.

5.2.2 The discrete Fourier transform (DFT)

The discrete Fourier transform (DFT) is covered in detail in Chapter 4. It is presented here because one important interpretation of the DFT is as frequency samples of the DTFT. As seen in the previous section, the discrete Fourier series (DFS) can also be interpreted as frequency samples of the DTFT. The difference between the two is that *the DFS applies to periodic signals, while the DFT applies to finite signals.* The DFS corresponding to a particular DFT pertains to the periodic signals formed by replicating the finite signals at regular intervals. Or, looking at it from the other direction, the DFT is defined as one period of the DFS.

Using the frequency sampled signals defined in the previous section, the DFT is defined as one period the DFS in both time and frequency on the interval $[0, N-1]$ with all other values to zero, that is

$$x_2[n] = \begin{cases} x_1[n] * p[n] & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$X_2[k] = \begin{cases} X_1[k] & k = 0, \ldots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

A representation of the DFT as one period of the DFS is shown in Fig. 3(e).

The usefulness of the DFT arises from the fact that, using digital computers, it is not generally possible to compute the DTFT of a signal, because we cannot represent the continuous-frequency variable f without performing an infinite number of operations. Instead, we can compute a finite number of *frequency samples* of the DTFT. We hope that, if the spacing between samples is sufficiently small, these frequency samples will provide a good representation of the entire frequency spectrum represented in the DTFT. This leads to the definition of the N-point DFT, $X[k]$, of a finite signal $x[n]$ of length N, where we take k samples of its DTFT $X(f)$ at intervals of $1/N$:

$$X[k] \stackrel{\triangle}{=} X(f)\big|_{f=\frac{k}{N}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}\Bigg|_{f=\frac{k}{N}} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad \text{for } 0 \le k \le N-1. \quad (5.10)$$

Because $X(f)$ is periodic with period 1, $X[k]$ is periodic with period N, which justifies only considering the values of $X[k]$ over the interval $[0, N-1]$.

5.2.3 The continuous-time Fourier series

In this section, we will show that the continuous-time Fourier series (CTFS) is a special case of the CTFT, based on sampling in frequency, in the same way that the DTFT is a special case of the CTFT, based on sampling in time. This is expected because time and frequency play symmetric roles in the CTFT, we have seen that the DTFT is a periodic, continuous function of frequency, and we have seen that the CTFS represents signals that are periodic and continuous in time.

To show this, consider the continuous-time, periodic signal $x(t)$ with period T. This signal can be considered as the convolution of one of its periods $x_T(t)$ by a periodic impulse train.

$$x(t) = x_T(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{5.11}$$

where

$$x_T(t) \triangleq \begin{bmatrix} x(t) & \text{if} \quad 0 \leq t \leq T \\ 0 & \text{otherwise} \end{bmatrix}$$

The CTFT of $x(t)$ is

$$X_T(F) = \int_{-\infty}^{\infty} x_T(t) e^{-j2\pi Ft} dt = \int_0^T x(t) e^{-j2\pi Ft} dt.$$

Using this expression, Eq. (3.20l) and the convolution theorem, we take the transform of Eq. (5.11) to show that $X(F)$ is a train of frequency impulses spaced at intervals of $\frac{1}{T}$, in other words that the CTFT of a periodic signal is discrete in frequency:

$$X(F) = X_T(F) \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(F - \frac{k}{T}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_T(\frac{k}{T}) \delta\left(F - \frac{k}{T}\right) = \sum_{k=-\infty}^{\infty} X_k \delta\left(F - \frac{k}{T}\right)$$

where we have defined

$$X_k \triangleq \frac{1}{T} X_T\left(\frac{k}{T}\right) = \frac{1}{T} \int_0^T x(t) e^{-j2\pi kt/T} dt \tag{5.12}$$

The X_k given in Eq. (5.12) are the Fourier series coefficients of the periodic signal $x(t)$. The signal $x(t)$ can be expressed as a function of the X_k by writing the inverse CTFT relation:

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF = \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} X_k \delta\left(F - \frac{k}{T}\right) \right] e^{j2\pi Ft} dF$$

Interchanging the orders of integration and summation, and applying the definition of the impulse, we obtain the Fourier series expansion of a periodic signal (CTFS), Eq. (1.A.10):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}.$$

We give without proof the convolution and product theorems for Fourier series:

$$\frac{1}{T} x(t) \circledast_T y(t) \longleftrightarrow X_k Y_k$$

$$x(t) y(t) \longleftrightarrow X_k * Y_k$$

To summarize, the DTFT and the CTFS are both special cases of the CTFT in which the roles of time and frequency are interchanged. This is a further example of the duality between the time and frequency domains.

5.3 Applications

5.3.1 Digital-filter implementation of continuous-time LTI systems

An important consequence of the sampling theorem is that digital filtering and continuous-time filtering are equivalent for band-limited signals. To show this, consider a continuous-time signal $x(t)$, and a continuous-time LTI system with impulse response $h(t)$. Assume that both $x(t)$ and $h(t)$ are bandlimited to W Hz.[1] We will show that the system output $y(t) = x(t) * h(t)$ can be obtained by the following sequence of operations (Fig. 4):

1. Form the discrete-time signal $x[n]$ by sampling $x(t)$ at a frequency $F_s > 2W$.
2. Sample the impulse response $h(t)$ at the same frequency to form $h[n]$.
3. Compute the discrete-time convolution $y[n] = x[n] * h[n]$.
4. Interpolate $y[n]$ using Eq. (5.3) to form the continuous-time signal $y_R(t)$.

It can be shown that

$$y_R(t) = F_s y(t)$$

To see this, it is easiest to think in the frequency domain (Fig. 5). Because we have assumed that $x(t)$ and $h(t)$ are bandlimited, their transforms are equal (within the multiplicative constant F_s) to the transforms of $x[n]$ and $h[n]$, respectively, over the range $\frac{-F_s}{2} \leq f \leq \frac{F_s}{2}$. Therefore, over that range, $Y(f)$, the DTFT of $y[n]$, which is the product of $H(f)$ and $X(f)$ is equal to the spectrum of $y(t)$ multiplied by F_s^2. Using the interpolation formula, $y_R(t)$, which is a lowpass filtered version of $y[n]$, is equal to $y(t)$ times F_s.

The result that continuous-time *linear* systems can be exactly implemented in discrete-time for bandlimited inputs does not hold for nonlinear systems. This is because nonlinear systems produce high-frequency distortion components that can be aliased even if sampling of the original signal verifies Nyquist's criterion. For example, Fig. 6 shows that the signal formed by interpolation of the output of a squarer implemented in discrete time is not equal to the output of a continuous-time squarer. Thus, discrete-time implementations of *nonlinear* continuous-time systems require oversampling the input signals.

5.3.2 Spectral analysis using the DFT

One major application of frequency sampling is spectral analysis, that is, analyzing the frequency content of signals. The analysis of an arbitrary discrete-time signal, $x[n]$, consists of two stages, first windowing and then computing the frequency samples, or DFT. These two operations are required because using a digital computer, we can only consider signals of finite-duration

[1] If $h(t)$ is not bandlimited, we can always form a new filter with a bandlimited impulse response by passing $h(t)$ through an ideal lowpass filter with cutoff frequency W. This will not change the response of the filter to bandlimited inputs.

and compute the frequency samples at finite intervals in frequency. Specifically, the windowing operation is

$$v[n] = x[n]w[n]$$

where $w[n]$ is a finite-duration window of length N and equal to zero outside the range $0 \leq n \leq N-1$. We obtain frequency samples of the DTFT, at $f = \frac{k}{N}$:

$$V[k] = V(f)|_{f=\frac{k}{N}} = \sum_{n=-\infty}^{\infty} v[n]e^{-j2\pi fn}\bigg|_{f=\frac{k}{N}} = \sum_{n=0}^{N-1} v[n]e^{-j2\pi kn/N} \qquad (5.13)$$

for $k = 0, \ldots, N-1$, where we have used definition of the DTFT and the fact $v[n]$ is of finite duration. Equation (5.13) corresponds to the DFT of the windowed signal $v[n]$, as defined by Eq. (5.10).

If the discrete-time signal $x[n]$ originally resulted from sampling a continuous-time signal (with sampling period T_s), then the frequency samples correspond to the continuous-time frequencies, F_k, given by

$$F_k = \frac{k}{NT_s}.$$

For simplicity, we will discuss discrete-time signals. However, the following discussion also applies to spectral analysis of bandlimited continuous-time signals that have been sampled appropriately.

In order to properly interpret results of our spectral analysis, we must understand the effects of both windowing and frequency sampling. To gain insight into these effects, we will study the effects of windowing and frequency sampling for a simple case, the sum of two cosines. We will consider windowing first.

Effect of windowing

Define the discrete-time signal

$$x[n] = A_0 \cos(2\pi f_0 n) + A_1 \cos(2\pi f_1 n)$$

Using (3.12f), its DTFT for the period $-\frac{1}{2} < f < \frac{1}{2}$ is

$$X(f) = \frac{A_0}{2}[\delta(f - f_0) + \delta(f + f_0)] + \frac{A_1}{2}[\delta(f - f_1) + \delta(f + f_1)].$$

We will use the rectangular window $w[n]$ of length N,

$$w[n] = \begin{cases} 1 & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

In the time domain, the windowed signal $v[n]$ is

$$v[n] = x[n]w[n] = \begin{cases} x[n] & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

In the frequency domain, the product theorem shows that this corresponds to cyclic convolution with $W(f)$, the DTFT of the rectangular window. Considering only one period on the interval $-\frac{1}{2} < f < \frac{1}{2}$ yields

$$V(f) = X(f) \circledast W(f) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(\phi) W(f-\phi)\, d\phi \tag{5.14}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{A_0}{2}[\delta(\phi - f_0) + \delta(\phi + f_0)] + \frac{A_1}{2}[\delta(\phi - f_1) + \delta(\phi + f_1)] \right) W(f-\phi)\, d\phi \tag{5.15}$$

$$= \frac{A_0}{2}[W(f - f_0) + W(f + f_0)] + \frac{A_1}{2}[W(f - f_1) + W(f + f_1)]. \tag{5.16}$$

The DTFT of the windowed signal consists of scaled replicas of the DTFT of the window at the frequencies of the original cosines.

The implications of the above analysis are best illustrated by an example. Specifically, we will consider a rectangular window of length $N = 64$, amplitudes $A_0 = 1$ and $A_1 = 0.75$, and frequency $f_0 = 0.1$. We will vary f_1 to examine the effects of windowing. Figure 7(a) shows the magnitude of the DTFT of a rectangular window of length $N = 64$. Figure 7(b) shows the magnitude of $V(f)$, the DTFT of the windowed sum of cosines, for $f_1 = 0.3$. As predicted above, the effect of windowing is to produce a scaled replica of the window's spectrum at the frequencies of the cosines ($f = \pm 0.1$ and $f = \pm 0.3$). Figure 7(c)—3(e) shows the magnitude of $V(f)$ for $f_1 = 0.15$, $f_1 = 0.12$, and $f_1 = 0.108$, respectively. As the frequency f_1 approaches f_0, the smearing of the spectrum caused by the window is more detrimental. In Fig. 7(e), the two distinct frequencies can no longer be resolved.

If a particular frequency resolution is desired, it can be obtained by using a sufficiently long window. Increasing the length of the window in time corresponds to narrowing the width of its main lobe in frequency. Figure 8 shows results obtained for the same example as Fig. 7, but with the window length increased to $N = 128$. Comparing Fig. 8(a) to Fig. 7(a) verifies that the longer window does have a narrower main lobe. This results in less smearing of the spectrum and in successful resolution of the two closely-spaced frequencies in Fig. 8(e).

Effect of frequency sampling

The second stage of spectral analysis consists of computing the frequency samples, that is, the DFT of the windowed signal. We are really interested in the DTFT of the signal, but due to computational constraints we are restricted to using the DFT to sample the DTFT at particular frequencies. This can produce misleading results, as the following example illustrates.

This example again uses the windowed sum of two cosines with amplitudes $A_0 = 1$ and $A_1 = 0.75$ and window length $N = 64$. Figure 9(c) shows the magnitude of the 64-point DFT for cosines of frequency $f_0 = \frac{1}{8}$ and $f_1 = \frac{3}{16}$. Figure 9(d) shows the magnitude of the 64-point DFT for cosines of frequency $f_0 = \frac{7}{48}$ and $f_1 = \frac{10}{48}$. Note that the frequency separation is the same in both examples ($f_1 - f_0 = \frac{1}{16}$). But the 64-point DFTs of the two signals are quite different. In Fig. 9(c), the DFT has strong spectral lines at the frequencies of the two cosines in the signal and no frequency content anywhere else, while in Fig. 9(d) the DFT exhibits significant spectral smearing, consistent with the previous example in Figs. 7 and 8. The very clean appearance of the DFT in Fig. 9(c) results from sampling the spectrum at particular locations. This can be understood by looking at the corresponding unsampled spectra, based on the DTFT, shown in Fig. 9(a) and (b). The choice of parameters in Fig. 9(c) resulted in sampling the spectrum at locations where it is exactly zero in Fig. 9(a).[2] This was not the case for the parameters selected in Fig. 9(d), as seen by comparing it to Fig. 9(b).

It is possible to avoid such misleading results by sampling the spectrum with sufficient resolution, that is, computing a sufficiently long DFT. The signal $v[n]$ was restricted to length N by the window, but it is possible to calculate the M-point DFT ($M > N$) by padding $v[n]$ to length M with zeros. Figure 9(e) and (f) shows the results using a longer DFT. The signals were created from the same sum of cosines and 64-point rectangular window used in Fig. 9(a) and (b). Then the 64-point signals were padded with zeros to length $M = 128$. The magnitude of the 128-point DFTs are shown in Fig. 9(e) and (f). The more closely spaced frequency samples clearly provide a more accurate visual representation of the underlying spectrum.

To summarize, the examples shown in Figs. 7–9 illustrate that the DFT can be used effectively for spectral analysis, providing that parameters are selected appropriately. The window length and the length of the DFT both affect the resulting frequency samples. In particular, the window length is related to the smearing of the spectrum, while the DFT length controls the spacing of the frequency samples.

5.4 Summary

In this chapter, we considered the operations of sampling in time and sampling in frequency. It was shown that sampling in one domain (multiplication by a periodic impulse train) corresponds to forming a periodic signal in the other domain (convolution with a periodic impulse train). Furthermore, we saw that the reconstruction of a continuous-time signal (convolution with a sinc function) corresponds to multiplying with the ideal lowpass filter in the frequency domain. Similarly, reconstruction of a frequency-sampled signal is accomplished by convolving the frequency samples with a sinc function in the frequency domain, which corresponds to multiplication with a rectangular window in the time domain.

The idea of sampling was used to show that the CTFS and the DTFT are both special cases of the CTFT, where the roles of time and frequency are interchanged. The CTFS, which pertains to signals that are periodic in time, is obtained by frequency sampling the CTFT. The DTFT, which pertains to signals that have been sampled in time, produces a representation that is periodic in frequency. This relationship between the CTFT and the DTFT for signals related by sampling in time is the basis for the important application of implementing continuous-time LTI systems with digital filters.

If the DTFT is subsequently sampled in frequency, we obtain the DFS, which pertains to signals that are discrete and periodic in time and frequency. Defining signals that are finite in both time and frequency, based on one period of the DFS signals, we obtain the DFT. The DFT was

[2] If done deliberately, such sampling of the spectrum at locations where it is exactly zero is a useful technique to eliminate some effects of the window. However, it requires prior knowledge of the frequency content of the windowed signal.

summarized here because of its important interpretation as frequency samples of the DTFT, but it is covered in more detail in Chapter 4. One important application of the DFT is spectral analysis, which allows us to determine the frequency content of finite signals using a digital computer.

Further Reading

Karu Chapter 19.
Oppenheim and Schafer: Chapter 4; Chapter 10, Sections 1 and 2.
Oppenheim, Wilsky and Nawab: Chapter 7.
Siebert Chapter 14.

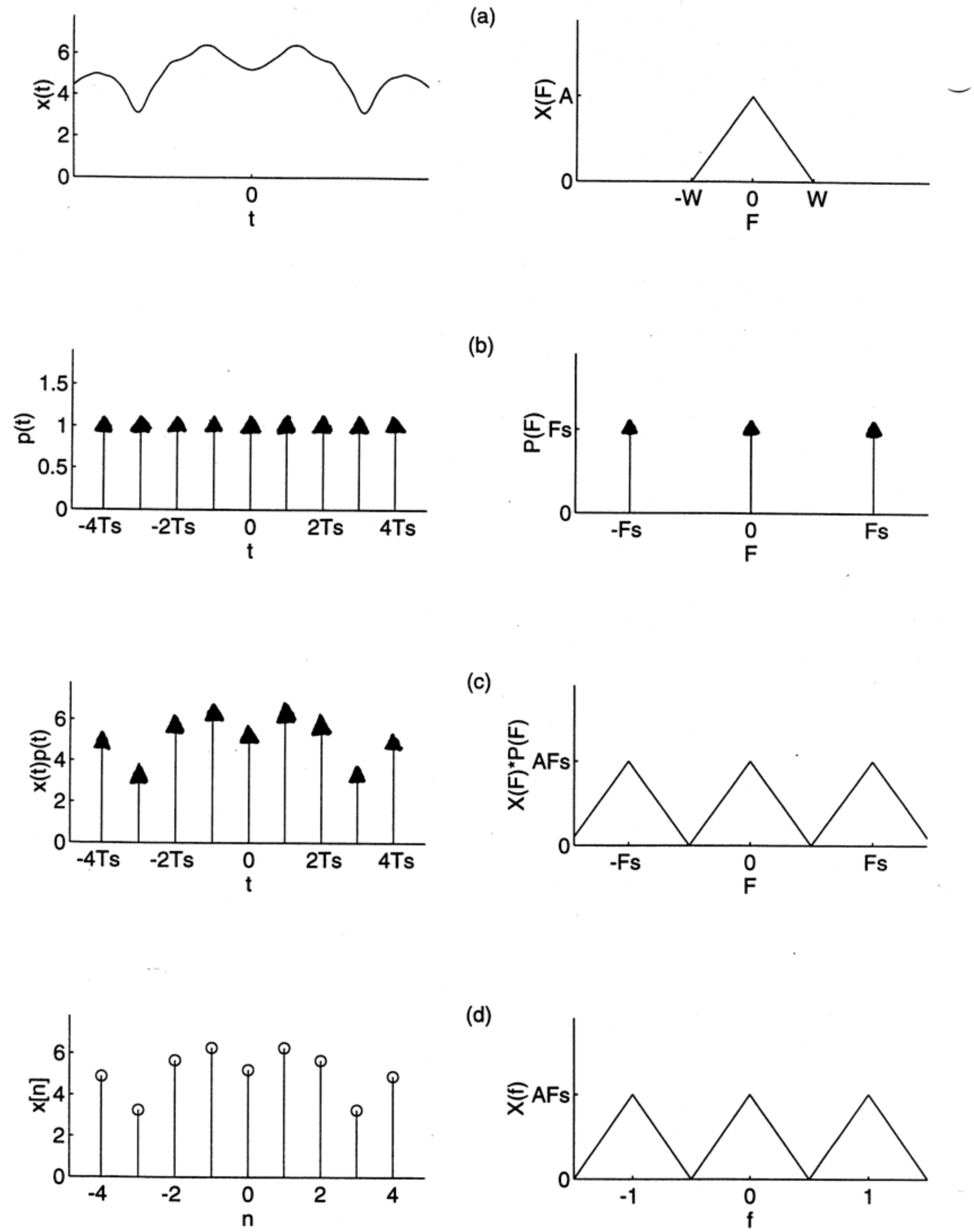

Figure 5.1: Sampling in time. (a) Continuous-time signal, bandlimited to W, and a representation of its CTFT. (b) The sampling function, $p(t)$, and its CTFT. (c) Sampled signal, $x_s(t) = x(t)p(t)$, and its CTFT, $Xs(F) = X(F) * P(F)$. (d) Discrete-time signal and its DTFT.

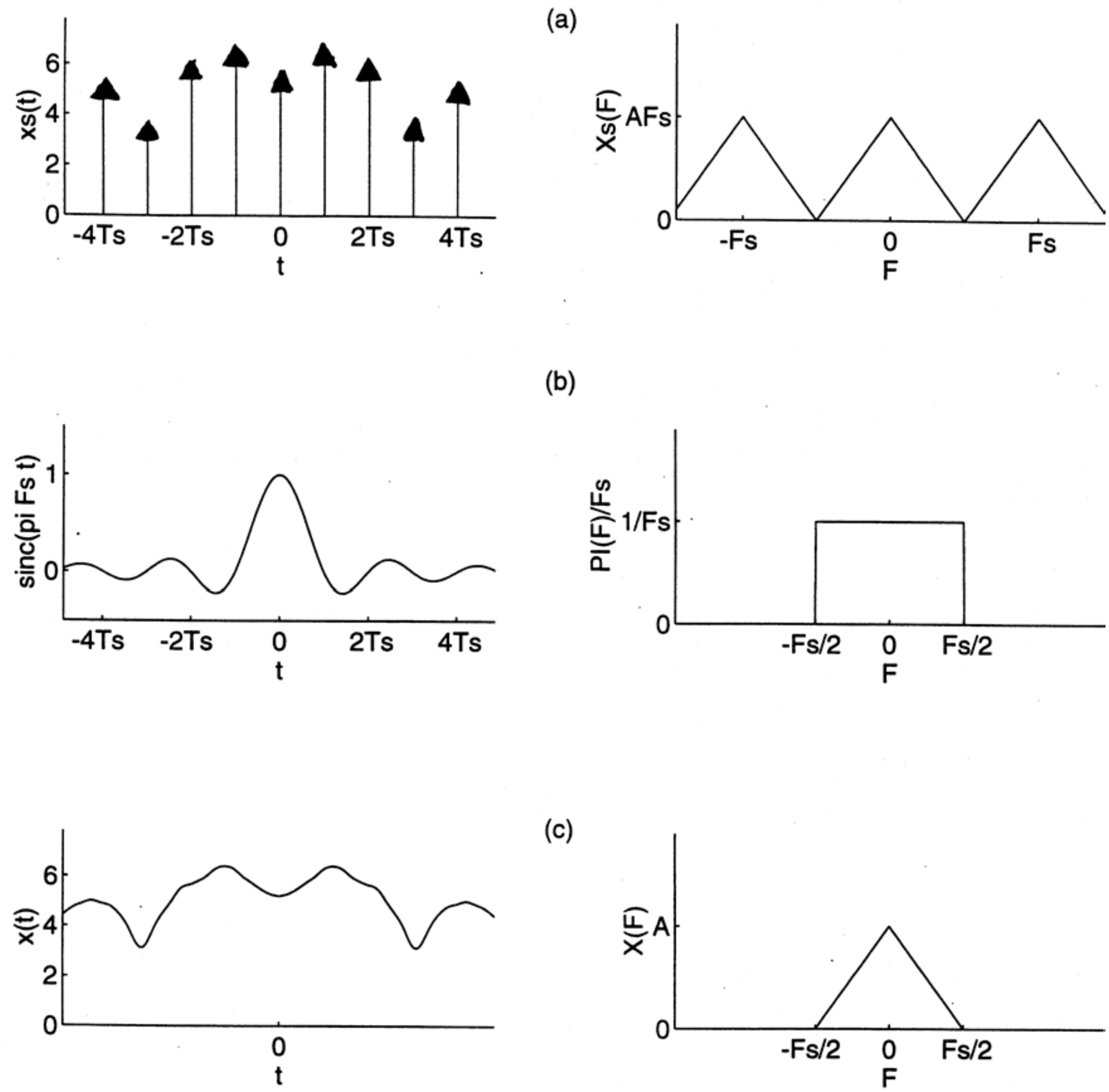

Figure 5.2: Reconstruction of a continuous-time signal from its samples. (a) Sampled signal and a representation of its CTFT. (b) The interpolation function in time, $sinc(\pi F_s t)$, corresponds to the ideal lowpass filter, $\frac{1}{F_s}\Pi_{\frac{F_s}{2}}(F)$, in frequency. The scale factor F_s is included in these functions for convenience. (c) The reconstructed signal, $x(t) = x_s(t) * sinc(\pi F_s t)$, and a representation of its CTFT, $X(F) = \frac{1}{F_s}X_s(F)\Pi_{F_s/2}(F)$.

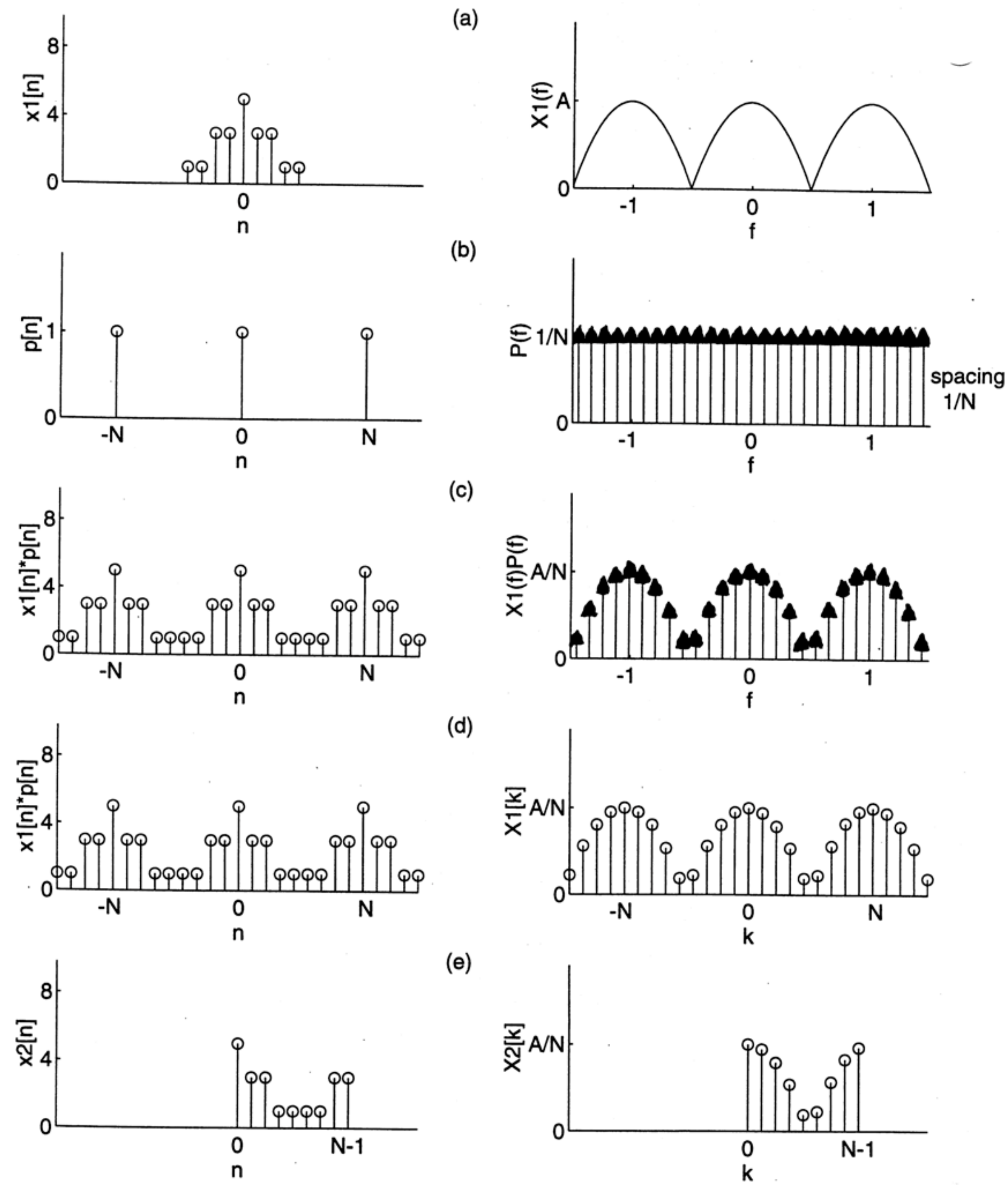

Figure 5.3: Sampling a discrete-time signal in frequency. (a) Discrete-time signal of duration N, and a representation of its DTFT. (b) The frequency sampling function. (c) Sampled frequency signal and the resulting periodic signal in the time domain. (d) Discrete-frequency representation corresponding to the DFS. (e) The DFT corresponds to one period DFS signals shown in (d), on the interval $[0, N-1]$.

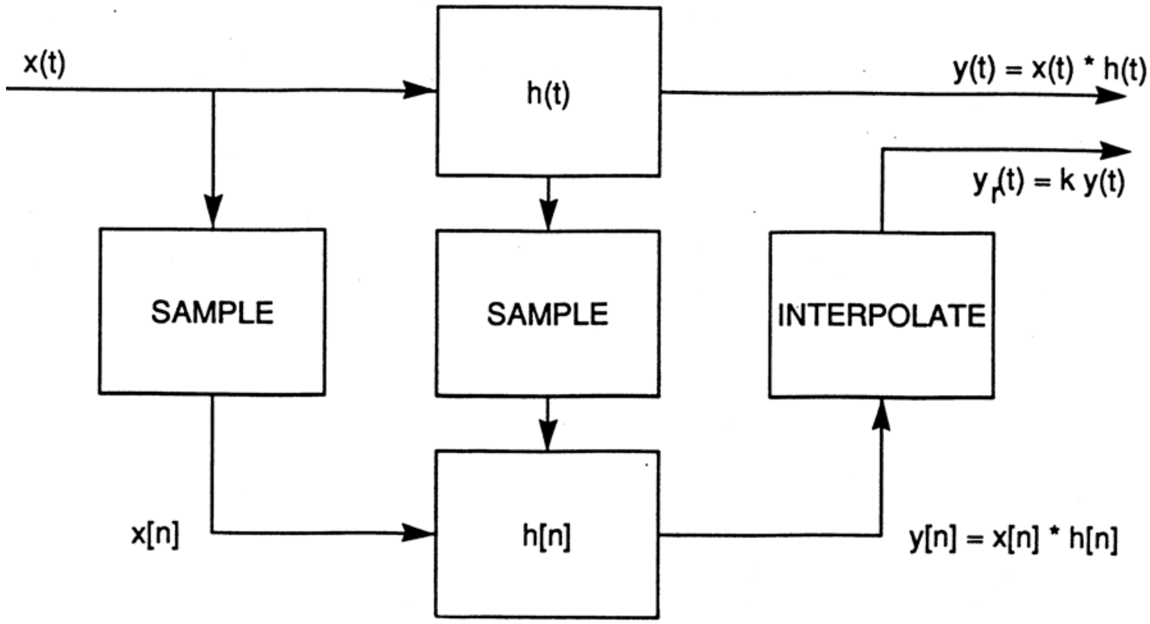

Figure 5.4: Digital-filter implementation of continuous-time linear systems.

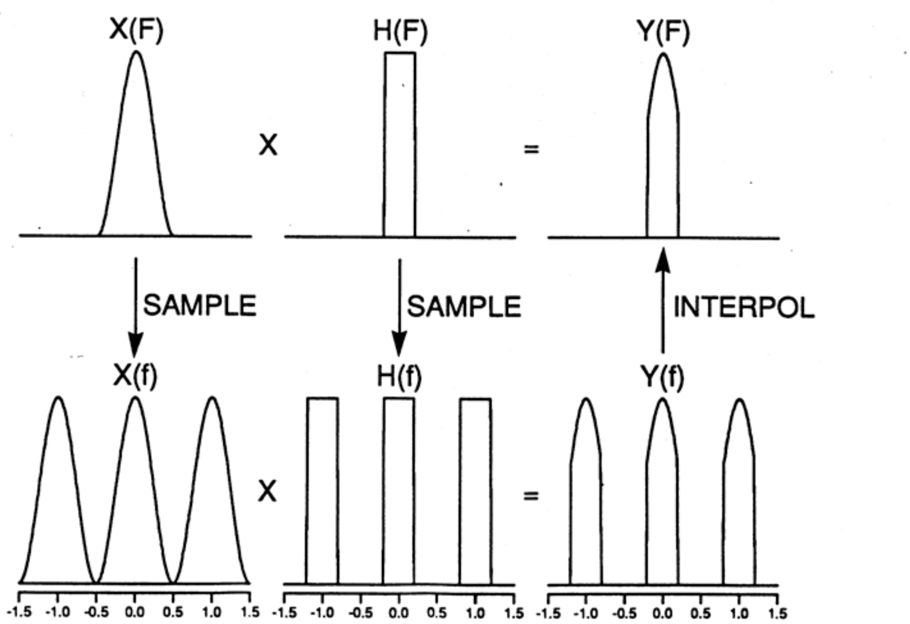

Figure 5.5: Frequency-domain representation of the digital-filter implementation of a continuous-time linear system.

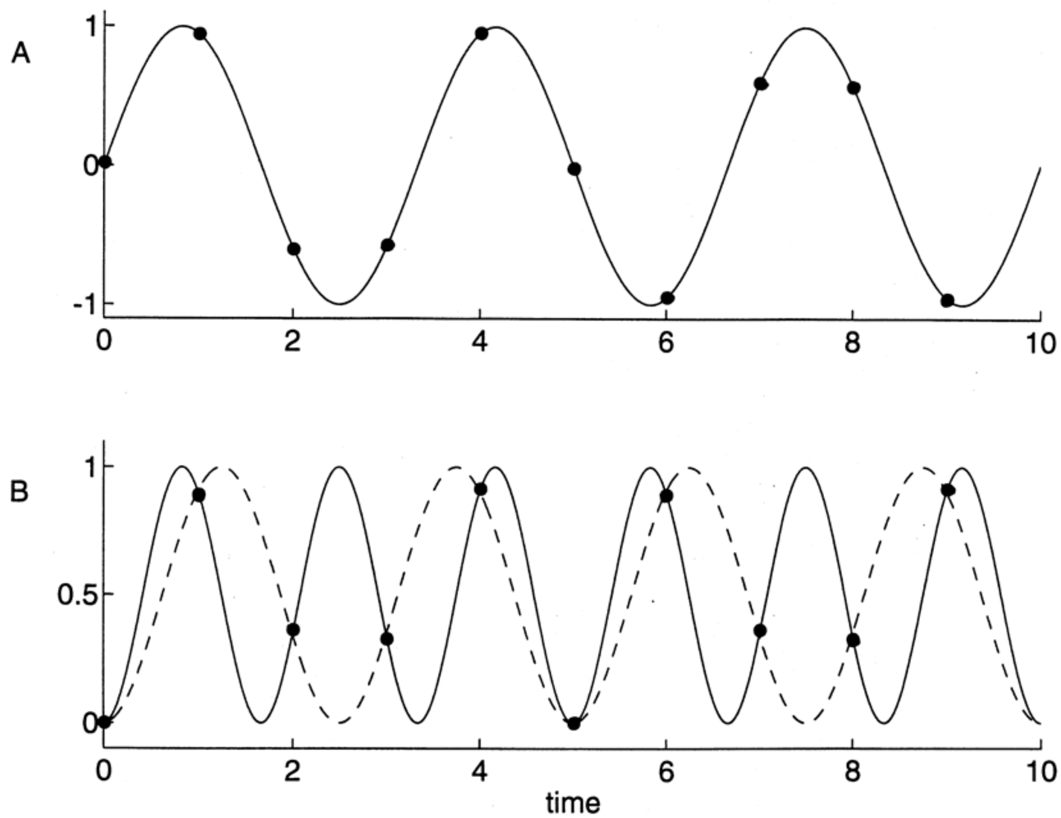

Figure 5.6: (A) Sine wave sampled at 3.33 times its frequency. (B) The solid line shows the sine wave squared. The dashed line shows that the square of the sampled signal is aliased to a lower frequency, even though the original signal was not.

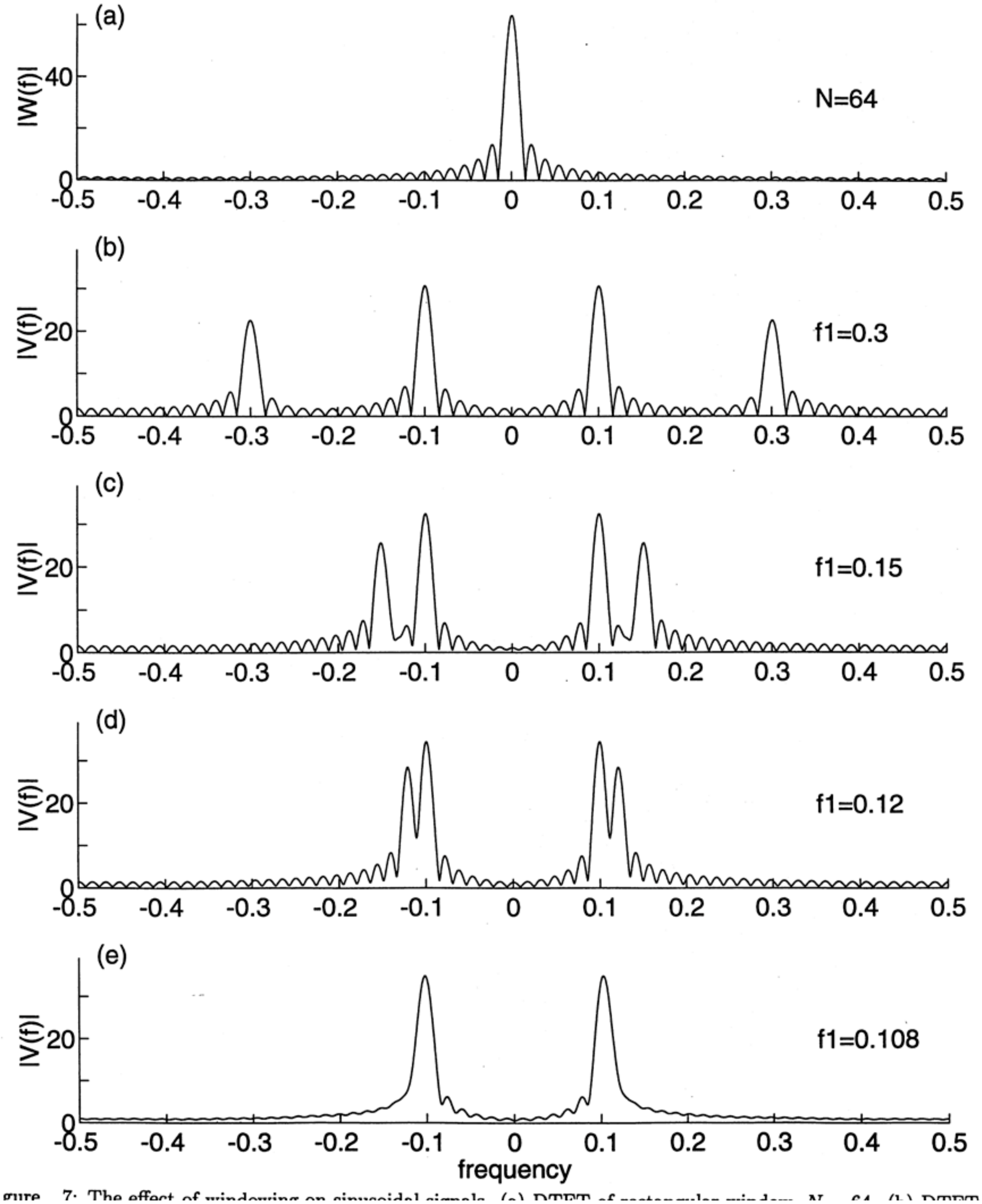

Figure 5.7: The effect of windowing on sinusoidal signals. (a) DTFT of rectangular window, $N = 64$. (b) DTFT of cosine sequence $A_0 \cos(2\pi f_0 n) + A_1 \cos(2\pi f_1 n)$ with $f_0 = 0.1$, $f_1 = 0.3$, $A_0 = 1$, and $A_1 = \frac{3}{4}$ for $n = 0, \ldots, 63$. (c) DTFT of cosine sequence as in (b) with $f_1 = 0.15$. (d) DTFT of cosine sequence as in (b) with $f_1 = 0.12$. (e) DTFT of cosine sequence as in (b) with $f_1 = 0.108$.

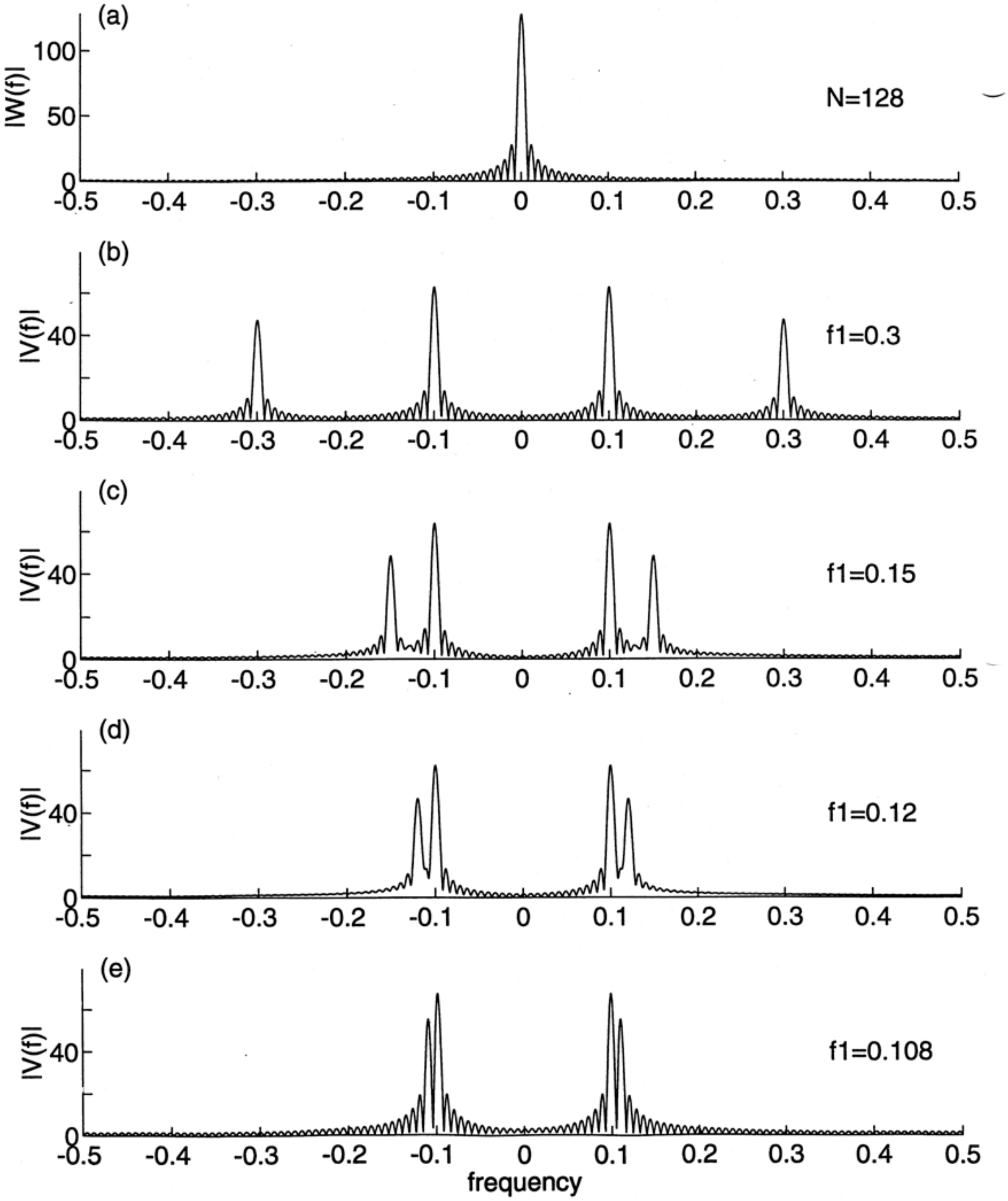

Figure 5.8: The effect of windowing on sinusoidal signals. (a) DTFT of rectangular window, $N = 128$. (b) DTFT of cosine sequence $A_0 \cos(2\pi f_0 n) + A_1 \cos(2\pi f_1 n)$ with $f_0 = 0.1$, $f1 = 0.3$, $A_0 = 1$, $A_1 = \frac{3}{4}$ for $n = 0, \ldots, 127$. (c) DTFT of cosine sequence as in (b) with $f_1 = 0.15$. (d) DTFT of cosine sequence as in (b) with $f_1 = 0.12$. (e) DTFT of cosine sequence as in (b) with $f_1 = 0.108$.

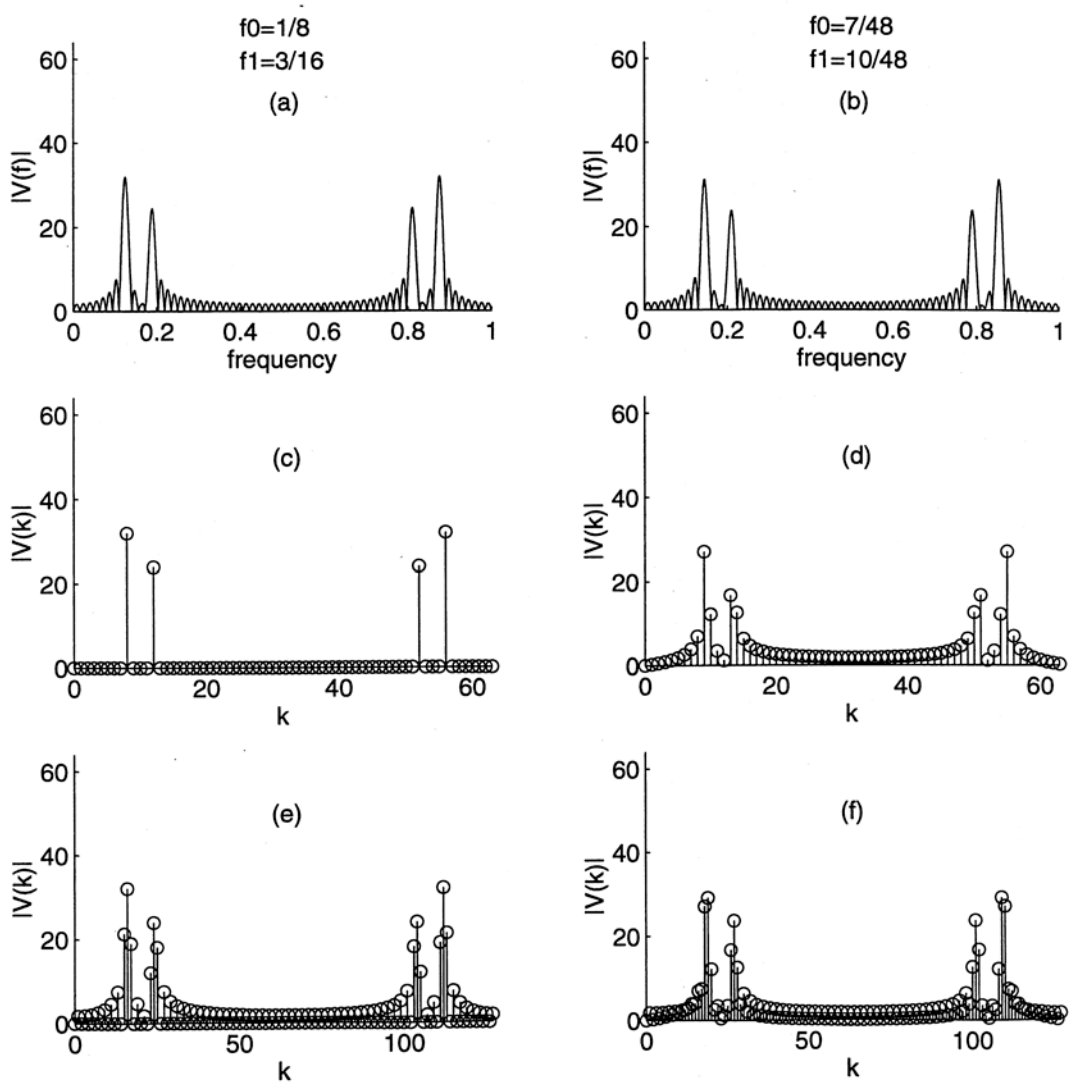

Figure 5.9: The effect of spectral sampling on sinusoidal signals with a 64-point rectangular window. (a) DTFT of cosine sequence $A_0 \cos(2\pi f_0 n) + A_1 \cos(2\pi f_1 n)$ with $f_0 = \frac{1}{8}$, $f_1 = \frac{3}{16}$, $A_0 = 1$, and $A_1 = \frac{3}{4}$ for $n = 0, \ldots, 63$. (b) DTFT of cosine sequence as in part (a) except $f_0 = \frac{7}{48}$ and $f_1 = \frac{10}{48}$. (c) 64-point DFT of cosine sequence in part (a). (d) 64-point DFT of cosine sequence in part (b). (e) 128-point DFT of cosine sequence in part (a). (f) 128-point DFT of cosine sequence in part (b).