Banach-Tarski: Preliminaries

1 The Theorem

Banach-Tarski Theorem It is possible to decompose a ball into a finite number of pieces and reassemble the pieces (without changing their size or shape) so as to get two balls, each of the same size as the original.

1.1 Warm-Up Case 1: A Line

It is possible to decompose $[0, \infty) - \{1\}$ into two distinct parts, and reassemble the parts (without changing their size or shape) so as to get back $[0, \infty)$.

- Decompose $[0, \infty) - \{1\}$ into: (i) $\{2, 3, 4, \ldots\}$ and (ii) everything else.
- Translate $\{2, 3, 4, \ldots\}$ one unit to the left.

1.2 Warm-Up Case 2: A Circle

It is possible to decompose $S^1 - \{p\}$ into two distinct parts, and reassemble the parts (without changing their size or shape) so as to get back $S^1$.

- Decompose $S^1 - \{p\}$ into: (i) $B$ and (ii) everything else.
- Rotate $B$ one unit counter-clockwise.
\[ B = \{ x \in S^1 : x \text{ is } n \text{ units clockwise from } p \ (n \in \mathbb{Z}^+) \} \]

The first six members of \( B \).

### 1.3 Warm-Up Case 3: The Cayley Graph

It is possible to decompose (the set of endpoints of) the Cayley Graph\(^1\) into four distinct parts, and reassemble the parts (albeit changing their size) so as to get back \emph{two copies} of the same size as the original.

\[ \text{Decompose } C^e \text{ into quadrants: } L^e, R^e, U^e, D^e. \]

\(^1\)A Cayley Path is a finite sequence of steps starting from \( c \), where no step follows its inverse. The Cayley Graph \( C \) is the set of Cayley Paths. \( X^e \) is the set of endpoints of Cayley paths in \( X \).
• Make first copy by expanding $R^e$ and translating left to meet $L^e$.
• Make second copy by expanding $U^e$ and translating down to meet $D^e$.

1.4 A more abstract description of the procedure

Notation: if $X$ is a set of Cayley Paths, let $\overrightarrow{X}$ be the set that results from eliminating the first step from each of the Cayley Paths in $X$.

By the definition of Cayley Paths:

$(\alpha) \quad C = \overrightarrow{R} \cup L$

$(\beta) \quad C = \overrightarrow{D} \cup U$

Since every Cayley Path has a unique endpoint, $(\alpha)$ and $(\beta)$ entail:

$(\alpha') \quad C^e = (\overrightarrow{R})^e \cup L^e$

$(\beta') \quad C^e = (\overrightarrow{D})^e \cup U^e$

On our two-dimensional interoperation of the Cayley Graph, this delivers the intended result because:

1. $C^e$ is decomposed into $U^e$, $D^e$, $L^e$ and $R^e$ (ignoring the central vertex)
2. One can get from $R^e$ to $(\overrightarrow{R})^e$, and from $D^e$ to $(\overrightarrow{D})^e$, by performing a translation together with an expansion.