Completeness of the SC Rules

We want to show that the system of rules is complete, in the sense that, whenever an argument is valid, the rules make it possible to derive the conclusion from the premisses. We intend to prove the following:

**Soundness and Completeness Theorem.** A sentence $\chi$ is a logical consequence of a set of sentences $\Gamma$ if and only if there is a derivation by which $\chi$ is derived from a premiss set that is included in $\Gamma$.

**Proof:** We've basically already proved the right-to-left direction, which is called the *soundness theorem*, since as we introduced the rules we verified that they ensured that, at each step in a derivation, the sentence we write down is a logical consequence of its premiss set. Our work is to prove the left-to-right direction, which is called the *strong completeness theorem*. To do this, we assume that there isn’t any derivation that infers $\chi$ from a premiss set that is included in $\Gamma$, and we show that there is a NTA under which all the members of $\Gamma$ are true and $\chi$ is false, by showing that there is a complete story that includes $\Gamma$ and leaves out $\chi$.

We are going to construct a set of sentences $\Gamma_\omega$ with the following three properties:

1. $\Gamma \subseteq \Gamma_\omega$.
2. $\chi$ isn’t derivable from $\Gamma_\omega$.
3. $\chi$ is derivable from each set that properly includes $\Gamma_\omega$.

We'll then verify that any set that satisfies conditions (2) and (3) is a complete story.

The construction of our set $\Gamma_\omega$ proceeds in stages, analogous to the construction in the compactness theorem. Whereas before we had make sure that, at the end of each stage, the set we had constructed had the property that each finite subset is consistent, now we need to ensure that, at every stage, the set we construct at that stage still has property (2). We begin by enumerating the sentences in an infinite sequence $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \ldots$. We now define a sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \ldots$ of sets of sentences, as follows:

$\Gamma_0 = \Gamma$.

By hypothesis, $\Gamma_0$ has property (2). Given $\Gamma_n$ with property (2), define:

$\Gamma_{n+1} = \Gamma_n \cup \{\zeta_n\}$, if that set has property (2).

$= \Gamma_n$, otherwise.

Let $\Gamma_\omega$ be the union of the $\Gamma_n$s. $\Gamma_\omega$ has property (1) because it contains $\Gamma_0$. If $\chi$ were derivable from $\Gamma_\omega$, it would be derivable from some finite subset of $\Gamma_\omega$, which means that it would be derivable from $\Gamma_n$, for sufficiently large $n$. Since $\chi$ isn’t derivable from any of the $\Gamma_n$s, it isn’t derivable from $\Gamma_\omega$ either. To verify (3), suppose that $\Delta \supseteq \Gamma_\omega$, and take $\psi$
\( \in \Delta \rightarrow \Gamma \). We can find \( k \) with \( \psi = \zeta_k \). Since \( \zeta_k \in \Gamma \), \( \zeta_k \notin \Gamma_{k+1} \), which can only happen if \( \chi \) is derivable from \( \Gamma_k \cup \{ \zeta_k \} \), and hence derivable from \( \Delta \).

If follows from (3) that, if \( \theta \) isn't an element of \( \Gamma \), then \( \chi \) must be derivable from \( \Gamma \cup \{ \theta \} \). If \( \theta \) were derivable from \( \Gamma \), then we could put the two derivations together to get a derivation of \( \chi \) from \( \Gamma \), contrary to (2). It follows that \( \Gamma \) is closed under deductive consequence, that is, that every sentence derivable from \( \Gamma \) is an element of \( \Gamma \). In particular, every SC theorem is in \( \Gamma \).

If \( \theta \) isn't in \( \Gamma \), then, by (3), \( \chi \) is derivable from \( \Gamma \cup \{ \theta \} \), which means, on account of rule CP, that \( (\theta \to \chi) \) is derivable from \( \Gamma \), and hence an element of \( \Gamma \).

To confirm that \( \Gamma \) is a NTA, we have to prove five clauses, one for each connective.

- **\( \neg \varphi \) is in \( \Gamma \) iff \( \varphi \) isn't in \( \Gamma \)**: The law of Duns Scotus tells us that \( (\neg \varphi \rightarrow (\varphi \rightarrow \chi)) \) is in \( \Gamma \), so that, if \( \Gamma \) contained both \( \varphi \) and \( \neg \varphi \), it would contain \( \chi \). If neither \( \varphi \) nor \( \neg \varphi \) were in \( \Gamma \), \( (\varphi \rightarrow \chi) \) and \( (\varphi \rightarrow \chi) \) would both be in \( \Gamma \). But then, since \( ((\varphi \rightarrow \chi) \rightarrow (\neg \varphi \rightarrow \chi)) \) is an SC theorem by TH9, \( \chi \) would be in \( \Gamma \).

- **\( (\varphi \land \psi) \) is in \( \Gamma \) iff \( \varphi \) and \( \psi \) are both in \( \Gamma \)**: Suppose that neither \( \varphi \) or \( \psi \) is in \( \Gamma \). Then \( (\varphi \rightarrow \chi) \) and \( (\psi \rightarrow \chi) \) are both in \( \Gamma \). By TH12, \( ((\varphi \lor \psi) \rightarrow \chi) \) is a logical consequence of \( ((\varphi \rightarrow \chi), (\psi \rightarrow \chi)) \), so that \( (\varphi \lor \psi) \) must not be in \( \Gamma \). By TH10 and TH11, \( (\varphi \lor \psi) \) is derivable both from \( \{\varphi\} \) and \( \{\psi\} \), so that, if either \( \varphi \) or \( \psi \) is in \( \Gamma \), so is their disjunction.

- **\( (\varphi \rightarrow \psi) \) is in \( \Gamma \) iff either \( \varphi \) isn't in \( \Gamma \) or \( \psi \) is in \( \Gamma \)**: If \( (\varphi \rightarrow \psi) \) and \( \varphi \) are both in \( \Gamma \), \( \psi \) is in \( \Gamma \) because \( \Gamma \) is closed under modus ponens. If \( \varphi \) isn't in \( \Gamma \), \( \neg \varphi \) is in \( \Gamma \). If \( \neg \varphi \) is in \( \Gamma \), so is \( (\varphi \rightarrow \psi) \), on account of TH6. If \( \psi \) is in \( \Gamma \), so is \( (\varphi \rightarrow \psi) \), on account of TH2.

- **\( (\varphi \lor \psi) \) is in \( \Gamma \) iff \( \varphi \) and \( \psi \) are both in \( \Gamma \) or neither of them is.** If \( \varphi \lor \psi \) is in \( \Gamma \), then, by DC and MP, \( ((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)) \) is in \( \Gamma \). It follows, using TH13 and TH14, that \( (\varphi \rightarrow \psi) \) and \( (\psi \rightarrow \varphi) \) are both in \( \Gamma \), so that, since \( \Gamma \) is closed under modus ponens, if \( \varphi \) is in \( \Gamma \), \( \psi \) is too, and in \( \psi \) is in \( \Gamma \), \( \varphi \) is too. If \( \varphi \) and \( \psi \) are both in \( \Gamma \), TH2 assures us that \( (\psi \rightarrow \varphi) \) and \( (\varphi \rightarrow \psi) \) are both in \( \Gamma \). It follows by TH15 that \( (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \) is in \( \Gamma \), from which it follows by DC that \( (\varphi \rightarrow \psi) \) is in \( \Gamma \). If, on the other hand, neither \( \varphi \) nor \( \psi \) is in \( \Gamma \), then both \( \neg \varphi \) and \( \neg \psi \) are in \( \Gamma \), and so, by TH6, \( (\varphi \lor \psi) \) is in \( \Gamma \). It follows, again by TH15 and DC, that \( (\varphi \lor \psi) \) is in \( \Gamma \).
Once we see that $\Gamma_\infty$ is a complete story, we get our NTA under which all the members of $\Gamma$ are true and $\chi$ is false by setting $\exists(\theta) = 1$ iff $\theta \in \Gamma_\infty$.

The strong completeness theorem immediately entails the so-called weak completeness theorem: Every valid SC sentence is an SC theorem.

The theorem provides an alternative proof of the compactness theorem. Suppose that $\Delta$ is inconsistent. Then $(P \land \neg P)$ is a logical consequence of $\Delta$, which implies, by the strong completeness theorem, that there is a derivation of $(P \land \neg P)$ from $\Delta$. The premiss set of the last line of this derivation is a finite subset of $\Delta$; call it $\Delta^*$. Since $(P \land \neg P)$ is derivable from $\Delta^*$, it follows by the soundness theorem that $(P \land \neg P)$ is a logical consequence. Hence $\Delta^*$ is a finite inconsistent subset of $\Delta$.

All in all, this is not a very impressive theorem. It succeeds in its advertised aim, which is to show that the particular system consisting of the five rules PI, CP, MP, MT, and DC, is sound and complete, but it doesn’t have a wider significance. It let’s us derive the compactness theorem, but we already knew the compactness theorem. It demonstrates that there is a proof procedure for demonstrating the validity of valid SC arguments, but the method of truth tables already gave us a much more powerful result, namely, that there is a decision procedure for testing the validity of SC arguments. There is a genuine benefit to proving the theorem, but it’s a benefit that won’t emerge until later: Proving the soundness and completeness theorem our deduction system for the sentential calculus will pave the way for the subsequent soundness and completeness theorem for the predicate calculus, where a proof procedure is precisely what one wants.