In most of these proofs, when things have got a little complicated, I have numbered the steps I’m taking in the hope that this makes things clearer. You aren’t required to do this when you answer, but I think it’s probably a good idea — it helps you get clear on what, exactly, you are doing, and it helps me understand what you are doing (if I can’t follow your proof, that’s bad).

Section 5.3E, Question 14

In what follows, I will refer to the following sentence:

- \( \Gamma \vdash P \) in SD if and only if \( \Gamma \models P \).

as ‘S&C’ (short for ‘Soundness and Completeness’ — you’ll see why I use this name in the near future).

Part (a)

Let \( \alpha \) be an argument of SL such that the set of assumptions that begin \( \alpha \) is \( \Gamma \) and the conclusion of \( \alpha \) is \( P \) (I’m using ‘\( \alpha \)’ so you don’t confuse it with a sentence letter of SL, but you can use whatever you like).

1. \( \alpha \) is valid in SD iff there is an SD derivation that has the members of \( \Gamma \) as primary assumptions and \( P \) in the scope of those assumptions only (by definition of ‘valid in SD’).

2. There is an SD derivation that has the members of \( \Gamma \) as primary assumptions and \( P \) in the scope of those assumptions only iff \( \Gamma \vdash P \) in SD (by definition of ‘\( \vdash \)’).

3. \( \Gamma \vdash P \) in SD iff \( \Gamma \models P \) (S&C).

4. \( \Gamma \models P \) iff there is no truth-value assignment such that every member of \( \Gamma \) is true and \( P \) is false (by definition of ‘\( \models \)’).

5. There is no truth-value assignment such that every member of \( \Gamma \) is true and \( P \) is false iff \( \alpha \) is truth-functionally valid (by definition of ‘truth-functionally valid’).

So, assuming S&C, an argument of SL is valid in SD if and only if the argument is truth-functionally valid.

Q.E.D.

Part (b)

1. A sentence \( P \) of SL is a theorem in SD iff \( \emptyset \vdash P \) in SD (by definition of theoremhood).

2. \( \emptyset \vdash P \) in SD iff \( \emptyset \models P \) (by S&C).
3. $\emptyset \models P$ iff there is no truth-value-assignment that makes every member of $\emptyset$ true and $P$ false (by definition of ‘$\models$’).

4. There is no truth-value-assignment that makes every member of $\emptyset$ true and $P$ false iff there is no truth-value assignment that makes $P$ false (as every truth-value assignment makes every member of $\emptyset$ true).

5. There is no truth-value assignment that makes $P$ false iff $P$ is truth-functionally true (definition of ‘truth-functionally true’).

So, assuming S&C, a sentence $P$ of SL is a theorem in SD if and only if $P$ is truth-functionally true.

Q.E.D.

Part (c)

1. Sentences $P$ and $Q$ of SL are equivalent in SD iff $\{P\} \vdash Q$ in SD and $\{Q\} \vdash P$ in SD (definition of ‘equivalent in SD’).

2. $P \vdash Q$ in SD and $Q \vdash P$ in SD iff $P \models Q$ and $Q \models P$ (by S&C).

3. $P \models Q$ and $Q \models P$ iff there is no truth-value assignment such that $P$ is true and $Q$ is false, and vice-versa. (by definition of ‘$\models$’).

4. There is no truth-value assignment such that $P$ is true and $Q$ is false, and vice-versa, iff $P$ and $Q$ are truth-functionally equivalent (by definition of ‘truth-functionally equivalent’).

So, assuming S&C, sentences $P$ and $Q$ of SL are equivalent in SD if and only if $P$ and $Q$ are truth-functionally equivalent.

Q.E.D.

Section 6.1E, Question 1

Part (b)

To show:

CLAIM: Every sentence of SL that contains no binary connectives is truth-functionally indeterminate.

CLAIM follows from the following…

BASIS CLAUSE: Every atomic sentence of SL is truth-functionally indeterminate.

INDUCTIVE STEP: If every sentence of SL containing (a) no binary connectives and (b) $n$ or fewer negations is truth-functionally indeterminate, then so is every sentence of SL containing no binary connectives and $n+1$ negations.
...as every sentence of SL that contains no binary connectives is a sentence of SL that contains no binary connectives and \( n \) negations, for some natural number \( n \).

The proof of Basis Clause is immediate — every atomic sentence of SL is such that there is a truth-value assignment that makes it true and a truth-value assignment that makes it false, so every atomic sentence of SL is truth-functionally indeterminate. It remains to prove Inductive Step.

**Proof of Inductive Step:**

1. Suppose every sentence \( P \) of SL containing (a) no binary connectives and (b) \( n \) or fewer negations is truth-functionally indeterminate (i.e., suppose the antecedent of Inductive Step).

2. Then for all such \( P \), there exists a truth-value assignment that make \( P \) true and a truth-value assignments that makes \( P \) false.

3. So, by the definition of \( \neg \), there is truth-value assignment that makes \( \neg P \) false and a truth-value assignment that make \( \neg P \) true, for all such \( P \).

4. But every sentence of SL containing no binary connectives and \( n + 1 \) negations is of the form \( \neg P \), for some such \( P \).

5. So for every sentence of SL containing no binary connectives and \( n + 1 \) negations, there is truth-value assignment that makes it true and a truth-value assignment that makes it false.

6. So every sentence of SL containing no binary connectives and \( n + 1 \) negations is truth-functionally indeterminate.

So, if every sentence of SL containing (a) no binary connectives and (b) \( n \) or fewer negations is truth-functionally indeterminate, then so is every sentence of SL containing no binary connectives and \( n + 1 \) negations.

Q.E.D.

**Part (e)**

Where \( P \) is a sentence of SL and \( Q \) is a sentential component of \( P \), let \([P](Q_1/\backslash Q)\) be a sentence that is the result of replacing at least one occurrence of \( Q \) in \( P \) with the sentence \( Q_1 \).

To show:

**Claim:** If \( Q \) and \( Q_1 \) are truth-functionally equivalent, then \( P \) and \([P](Q_1/\backslash Q)\) are truth-functionally equivalent.

Clearly, Claim follows from the following...
**Inductive Step:** If Claim is true for all $P$ containing $n$ or fewer connectives, it is true for all $P$ containing $n + 1$ connectives.

\[ \ldots \text{as every sentence of SL contains } n \text{ connectives, for some natural number } n. \]

**Proof of Basis Clause:**

1. $[P](Q_1//P)$ is just $Q_1$.
2. So if $Q_1$ is truth-functionally equivalent to $P$, then obviously $[P](Q_1//P)$ is truth-functionally equivalent to $P$.
3. But when $P$ is atomic, it’s only sentential component is $P$.
4. So, for all sentential components $Q$ of $P$, if $Q$ and $Q_1$ are truth-functionally equivalent, then $P$ and $[P](Q_1//Q)$ are truth-functionally equivalent, when $P$ is atomic.

So Claim is true when $P$ is atomic.

Q.E.D.

**Proof of Inductive Step:** Every sentence of SL containing $n + 1$ connectives is either of the form $\sim P^q$, for some $P$ containing $n$ connectives, or of the form $P \cdot R$, where $P, R$ contain $n$ or fewer connectives (and $\cdot$ is a variable that ranges over binary connectives of SL). I prove Inductive Step for each case in turn.

**Case 1:** Consider a sentence of SL of the form $\sim P^q$, where $P$ is a sentence containing $n$ connectives. Every sentential component $Q$ of $\sim P^q$ is either

(a) $\sim P^q$ itself, or
(b) is a sentential component of $P$.

I prove each sub-case in turn.

**Sub-case (a):** When $Q$ is $\sim P^q$ itself, $[\sim P^q](Q_1//Q)$ is truth-functionally equivalent to $\sim P^q$, for $Q_1$ truth-functionally equivalent to $Q$, by the same argument as in the proof of Basis Clause.

**Sub-case (b):**

1. Suppose $Q$ is a sentential component of $P$.
2. Suppose, also, that $P$ and $[P](Q_1//Q)$ are truth-functionally equivalent when $Q_1$ truth-functionally equivalent to $Q$ (i.e., suppose the antecedent of Inductive Step for the case of $P$).
3. Then $\sim P^q$ and $\sim ([P](Q_1//Q))^q$ are truth-functionally equivalent (by the definition of $\sim$).
4. And $\sim ([P](Q_1//Q))^q$ is identical to $[\sim P^q](Q_1//Q)$, when $Q$ is a sentential component of $P$. 

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5. So, if \( P \) and \([P](Q_1//Q)\) are truth-functionally equivalent, then \( \lnot \sim P \) is truth-functionally equivalent to \( \lnot \sim P \)(Q_1//Q), for \( Q_1 \) truth-functionally equivalent to \( Q \), when \( Q \) is a sentential component of \( P \).

So, if CLAIM is true for a sentence \( P \) containing \( n \) connectives, it is true for \( \lnot \sim P \). That concludes the proof for Case 1.

**Case 2:** Consider a sentence of SL of the form \( P \cdot R \), where \( P, R \) are sentences containing \( n \) or fewer connectives. Every sentential component of \( P \cdot R \) is either

(a) \( P \cdot R \) itself,

(b) a sentential component of \( P \) or a sentential component of \( R \) (or both).

I prove each sub-case in turn.

**Sub-case (a):** The proof here is the same as the proof of BASIS CLAUSE and sub-case (a) of Case 1, mutatis-mutandis.

**Sub-case (b):**

1. Suppose \( Q \) a sentential component of \( P \) or \( R \) or both.

2. Suppose, also, that \( P \) and \([P](Q_1//Q)\) are truth-functionally equivalent, and \( R \) and \([R](Q_1//Q)\) are truth-functionally equivalent, when \( Q_1 \) is truth-functionally equivalent to \( Q \) (i.e., suppose the antecedent of INDUCTIVE STEP for the cases of \( P \) and \( R \)).

3. Then \( P \cdot R \) is truth-functionally equivalent to \([P](Q_1//Q) \cdot [R](Q_1//Q)\) (by the relevant binary-connective's definition).

4. And \([P](Q_1//Q) \cdot [R](Q_1//Q)\) is identical to \([P \cdot R](Q_1//Q)\), when \( Q \) is a sentential component of \( P \) or \( R \).

5. So, if \( P \) and \([P](Q_1//Q)\) are truth-functionally equivalent, and \( R \) and \([R](Q_1//Q)\) are truth-functionally equivalent, then \( P \cdot R \) is truth-functionally equivalent to \([P \cdot R](Q_1//Q)\), for \( Q_1 \) truth-functionally equivalent to \( Q \), when \( Q \) is a sentential component of \( P \) or \( R \).

So, if CLAIM is true for sentences \( P, R \) containing \( n \) or fewer connectives, it is true for \( P \cdot R \).

That concludes the proof for Case 2.

So that concludes the proof for INDUCTIVE STEP.

Q.E.D.