Question 1

The trickiest things about this question is just stating what you want to prove in the right way.

Let \( Q_1, Q_2, \ldots, Q_n \) be atomic sentences. Let \( v(R) \) be the truth value assigned to \( R \) by truth-value assignment \( v \). Let \( \gamma(\ldots(P_1 \& P_2) \& \ldots \& P_n) \) be the sentence such that \( P_1 = Q_i \) iff \( v(Q_i) = T \), \( P_1 = \neg Q_i \) iff \( v(Q_i) = F \).

Showing that a sentence constructed in accordance with the characteristic sentence algorithm in TLB is, indeed, a characteristic sentence for the row of the truth-function schema in question is equivalent to showing that a truth-value assignment makes \( \gamma(\ldots(P_1 \& P_2) \& \ldots \& P_n) \) true if and only if that truth-value assignment assigns truth-values to the relevant atomic sentences in the way that \( v \) does. (You should think of \( v \) as the truth-value assignment that assigns truth-values in the way indicated by the row of the truth-function schema in question.) Another way of saying this: \( u(\gamma(\ldots(P_1 \& P_2) \& \ldots \& P_n)) = T \) iff \( u(Q_i) = v(Q_i) \), for all \( i \).

So what we want to show is that \( u(\gamma(\ldots(P_1 \& P_2) \& \ldots \& P_n)) = T \) iff \( u(Q_i) = v(Q_i) \), for all \( i \).

Here’s a proof:

1. By the fact in 6.1E (1d), \( u(\gamma(\ldots(P_1 \& P_2) \& \ldots \& P_n)) = T \) iff \( u(P_1) = u(P_2) = \ldots = u(P_n) = T \).

2. \( u(P_1) = u(P_2) = \ldots = u(P_n) = T \) iff both
   (a) for all \( i \) such that \( P_1 = Q_i \), \( u(P_1) = T \), and
   (b) for all \( i \) such that \( P_1 = \neg Q_i \), \( u(P_1) = T \)
   (as all \( i \) are either such that \( P_1 = Q_i \), or such that \( P_1 = \neg Q_i \)).

3. Now, 2(a) is true iff for all \( P_1 = Q_i \), \( u(Q_i) = T \), obviously, and 2(b) is true iff for all \( i \) such that \( P_1 = \neg Q_i \), \( u(Q_i) = F \) (by the definition of \( \neg \)).

4. So \( u(P_1) = u(P_2) = \ldots = u(P_n) = T \) iff both
   (a) for all \( P_1 = Q_i \), \( u(Q_i) = T \)
   (b) for all \( i \) such that \( P_1 = \neg Q_i \), \( u(Q_i) = F \).
   (by 2, 3).

5. But for all \( i \) such that \( P_1 = Q_i \), \( v(Q_1) = T \), and for all \( i \) such that \( P_1 = \neg Q_i \), \( v(Q_1) = F \) (by the definition of \( v \) above).

6. So \( u(P_1) = u(P_2) = \ldots = u(P_n) = T \) iff, \( u(Q_i) = v(Q_i) \), for all \( i \) (from 4, 5).

7. So \( u(\gamma(\ldots(P_1 \& P_2) \& \ldots \& P_n)) = T \) iff \( u(Q_i) = v(Q_i) \) (from 1, 6).

Q.E.D.
Question 2

Let $\Gamma$ be the set of atomic sentences.

Let $L_1$ be the set of all sentences $P$ such that $P \in L_1$ iff

(a) $P \in \Gamma$, or

(b) $P$ is of one of the following forms:

(i) $\neg Q$;

(ii) $Q \land R$;

(iii) $Q \lor R$;

where $Q, R \in L_1$.

Let $L_2$ be the set of all sentences $P$ such that

(a) $P \in \Gamma$, or

(b) $P$ is of the form $Q \rightarrow R$, where $Q, R \in L_2$.

We know that \{'\neg', '\&', '\lor'\} is TF-complete, so we know that for every truth-function, there is a sentence in $L_1$ that expresses that truth-function. If, for every sentence $P \in L_1$, there is a sentence in $L_2$ that expresses the same truth function as $P$, then it follows that for every truth-function, there is a sentence in $L_2$ that expresses that truth-function, from which it follows that \{'\rightarrow'\} is TF-complete.

So, all that remains to be done to prove that \{'\rightarrow'\} is TF-complete is to prove that for every sentence $P \in L_1$, there is a sentence in $L_2$ that expresses the same truth function as $P$.

I prove this by mathematical induction.

**Basis Clause:** For every atomic sentence $P \in L_1$ there is a sentence in $L_2$ that expresses the same truth function as $P$.

**Inductive Step:** If, for every sentence $P \in L_1$ containing $n$ or fewer connectives there is a sentence in $L_2$ that expresses the same truth function as $P$, then for every sentence $Q \in L_1$ containing $n + 1$ connectives there is a sentence in $L_2$ that expresses the same truth function as $L_2$.

Clearly, it follows from **Basis Clause** and **Inductive Step** that for every sentence $P \in L_1$, there is a sentence in $L_2$ that expresses the same truth function as $P$.

The proof of **Basis Clause** is immediate; for every atomic sentence $P \in L_1$ there is a sentence in $L_2$ that expresses the same truth function as $P$ — namely, $P$. (Besides, every atomic sentence expresses the same truth-function: the one that maps true to true and false to false.)

Now for **Inductive Step**. Consider an arbitrary sentence $Q \in L_1$ containing $n + 1$ connectives. $Q$ is either of the form
(a) $\sim R$, or
(b) $R \& S$, or
(c) $R \lor S$.

I prove inductive step for each case in turn.

**Case (a):**

1. Suppose that, for every sentence $P \in L_1$ containing $n$ or fewer connectives,
   there is a sentence in $L_2$ that expresses the same truth-function as $P$ (i.e.,
   suppose the antecedent of inductive step).
2. Then there is a sentence in $L_2$ that expresses the same truth-function as $R$ (as $R$ contains $n$ connectives).
3. I claim that $T \downarrow T$ expresses the same truth-function as $\sim R$. Here’s a sub-proof of my claim:
   (i) A truth-value assignment makes $\sim R$ true iff it makes $R$ false (by definition of ‘$\sim$’).
   (ii) A truth-value assignment makes $R$ false iff it makes $T$ false (by 2).
   (iii) A truth-value assignment makes $T$ false iff it makes $T \downarrow T$ true
        (by the definition of ‘$\downarrow$’. I leave the verification of this as an exercise for the reader).
   (iv) So a truth-value assignment makes $\sim R$ true iff it makes $T \downarrow T$ true.
   (v) So $\sim R$ expresses the same truth-function as $T \downarrow T$.
4. $T \downarrow T \in L_2$ (by the definition of $L_2$).
5. So there exists a sentence of $L_2$ that expresses the same truth-function as $\sim R$.
6. So, assuming the antecedent of inductive step, for every sentence of the form $\sim R$, where $R$ has $n$ connectives,
   there is a sentence in $L_2$ that expresses the same truth-function as $\sim R$.

That proves inductive step for case (a).

**Case (b):**

1. Suppose that, for every sentence $P \in L_1$ containing $n$ or fewer connectives,
   there is a sentence in $L_2$ that expresses the same truth-function as $P$ (i.e.,
   suppose the antecedent of inductive step).
2. Then there is a sentence in $L_2$ that expresses the same truth-function as $R$ and a sentence of $L_2$ that expresses the same truth-function as $S$ (as both $R$ and $S$ contain $n$ or fewer connectives). Call those sentence in $L_2$ $T$ and $U$, respectively.
3. I claim that \( \neg (T \downarrow T) \downarrow (U \downarrow U) \)\(^\neg\) expresses the same truth-function as \( \neg R \& S \). Here’s a sub-proof of my claim:

(i) A truth-value assignment makes \( \neg R \& S \) true iff it makes both \( R \) and \( S \) true (by definition of \( \& \)).

(ii) A truth-value assignment makes both \( R \) and \( S \) true iff it makes both \( T \) and \( U \) true and it (by 2).

(iii) A truth-value assignment makes both \( T \) and \( U \) true iff it makes \( \neg (T \downarrow T) \downarrow (U \downarrow U) \) true (by the definition of \( \downarrow \)). I leave the verification of this as an exercise for the reader.

(iv) So a truth-value assignment makes \( \neg R \& S \) true iff it makes \( \neg (T \downarrow T) \downarrow (U \downarrow U) \) true.

(v) So \( \neg R \& S \) expresses the same truth-function as \( \neg (T \downarrow T) \downarrow (U \downarrow U) \).

4. \( \neg (T \downarrow T) \downarrow (U \downarrow U) \in L_2 \) (by the definition of \( L_2 \)).

5. So there exists a sentence in \( L_2 \) that expresses the same truth-function as \( \neg R \& S \).

6. So, assuming the antecedent of \text{INDUCTIVE STEP}, for every sentence of the form \( \neg R \& S \), where \( R, S \) have \( n \) or fewer connectives, there is a sentence in \( L_2 \) that expresses the same truth-function as \( \neg R \& S \).

That proves \text{INDUCTIVE STEP} for case (b).

\textbf{Case (c)}:

1. Suppose that, for every sentence \( P \in L_1 \) containing \( n \) or fewer connectives there is a sentence in \( L_2 \) that expresses the same truth-function as \( P \) (i.e., suppose the antecedent of \text{INDUCTIVE STEP}).

2. Then there is a sentence in \( L_2 \) that expresses the same truth-function as \( R \) and a sentence of \( L_2 \) that expresses the same truth-function as \( S \) (as both \( R \) and \( S \) contain \( n \) or fewer connectives). Call those sentence in \( L_2 \) \( T \) and \( U \), respectively.

3. I claim that \( \neg (T \downarrow U) \downarrow (T \downarrow U) \)\(^\neg\) expresses the same truth-function as \( \neg R \lor S \). Here’s a sub-proof of my claim:

(i) A truth-value assignment makes \( \neg R \lor S \) true iff it does not make both \( R \) and \( S \) false (by definition of \( \lor \)).

(ii) A truth-value assignment does not make both \( R \) and \( S \) false iff it does not make both \( T \) and \( U \) false and it (by 2).

(iii) A truth-value assignment does not make both \( T \) and \( U \) false iff it makes \( \neg (T \downarrow U) \downarrow (T \downarrow U) \) true (by the definition of \( \downarrow \)). I leave the verification of this as an exercise for the reader.)
(iv) So a truth-value assignment makes \( \text{if} \ (R \lor S) \) true iff it makes \( \text{if} \ (T \downarrow U) \) \( \downarrow (T \downarrow U)^\sim \) true.

(v) So \( \text{if} \ (R \lor S) \) expresses the same truth-function as \( \text{if} \ (T \downarrow U) \downarrow (T \downarrow U)^\sim \).

4. \( \text{if} \ (T \downarrow U) \downarrow (T \downarrow U)^\sim \in L_2 \) (by the definition of \( L_2 \)).

5. So there exists a sentence in \( L_2 \) that expresses the same truth-function as \( \text{if} \ (R \& S) \).

6. So, assuming the antecedent of **Inductive Step**, for every sentence of the form \( \text{if} \ (R \lor S) \), where \( R, S \) have \( n \) or fewer connectives, there is a sentence in \( L_2 \) that expresses the same truth-function as \( \text{if} \ (R \lor S) \).

That proves **Inductive Step** for case (c).

And that finishes the proof of **Inductive Step**

And that finishes the proof that for every sentence \( P \in L_1 \), there is a sentence in \( L_2 \) that expresses the same truth function as \( P \).

And that finishes the proof that \( \{ \text{if} \} \) is TF-complete.

Q.E.D.

**Question 3**

There are, in fact, two ways to use the proof that SD is sound to prove that \( \text{SD}^* \) is sound. One way is to show that if \( \Gamma \vdash P \) in \( \text{SD}^* \), then \( \Gamma \vdash P \) in SD, so, by the soundness of SD, if \( \Gamma \vdash P \) in \( \text{SD}^* \), then \( \Gamma \models P \) — i.e., \( \text{SD}^* \) is sound.

I’m going to do it the other way, which is the way I think the question prompt is nudging you. The other way goes as follows.

The goal is to prove that \( \text{SD}^* \) is sound; i.e., that...

**SD* Soundness:** If \( \Gamma \vdash P \) in \( \text{SD}^* \), then \( \Gamma \models P \)

...is true. We are going to prove it by mathematical induction. Let \( P_k \) be the \( k \)th sentence in an \( \text{SD}^* \) derivation, and let \( \Gamma_k \) be the set of assumptions open on the \( k \)th line of that \( \text{SD}^* \) derivation. Clearly, **SD* Soundness** follows from the following:

**Basis Clause:** \( \Gamma_1 \models P_1 \).

**Inductive Step:** If \( \Gamma_{k+1} \vdash P_{k+1} \) in \( \text{SD}^* \), then \( \Gamma_{k+1} \models P_{k+1} \).

The proof of **Base Case** is exactly the same as the proof for the analogous base case in TLB on page 260, which is part of the proof of the soundness of SD. The proof of **Inductive Step** is exactly the same as the proof of the analogous inductive step on pages 260-264 of TLB, which is also part of the proof of soundness of SD, except you need to add a case for Backwards Conditional Introduction.

One way for that case to go is as follows:
**Case 13**: $P_{k+1}$ is justified by Backwards Conditional Introduction:

\[
\begin{array}{c|c|c}
  h & \sim R & \\
  j & \sim Q & \\
  k + 1 & R \supset Q (= P_{k+1}) & h \cdot j \cdot B \supset I \\
\end{array}
\]

1. By exactly the same proof as in case 8, we know that $\Gamma_{k+1} \models \lnot \sim R \supset \sim Q$.

2. So there is no truth-value assignment such that every member of $\Gamma_{k+1}$ is true and $\lnot \sim R \supset \sim Q$ is false (by the definition of $\models$).

3. So there is no truth-value assignment such that every member of $\Gamma_{k+1}$ is true and $\lnot \sim R$ is true and $\lnot \sim Q$ is false (by the definition of $\supset$).

4. So there is no truth-value assignment such that every member of $\Gamma_{k+1}$ is true and $R$ is false and $Q$ is true (by the definition of $\lnot$).

5. So there is no truth-value assignment such that every member of $\Gamma_{k+1}$ is true and $Q \supset R$ is false (again by the definition of $\supset$).

6. So $\Gamma_{k+1} \models Q \supset R$.

**So Inductive Step** is true in the case when $P_{k+1}$ is justified by Backwards Conditional Introduction. So, by that and the stuff in the proof of the soundness of $SD$, Inductive Step is true. So, by that and Base Case, $SD^*$ is sound. Q.E.D.