Logic – Session 18
Applying our formal semantics

Let I be the following interpretation:
\[ \text{UD} = \{a, b\} \quad \text{F} : \{<a>\} \quad \text{G} : \{<b>\} \]

Show that \( (\forall x)Fx \) is false on I.

\(<b> \notin I(F) \)

So for arbitrary d, \(<\text{den}_{I,d}[b/x](x)\> \notin I(F).\)

So by (2.), d[b/x] doesn't satisfy Fx on I.

So not: for every u \(\in\) UD, d[u/x] satisfies Fx on I.

So by (8.), not: d satisfies \((\forall x)Fx.\)

So not every variable assignment satisfies \((\forall x)Fx.\)

So by def. of truth, \((\forall x)Fx\) is false on I.
Show: $(\exists x)(Fx \supset (\forall y)Gy)$ is true on $I$

$(\exists x)(Fx \supset (\forall y)Gy)$ is true on $I$ iff every $d$ for $I$ satisfies $(\exists x)(Fx \supset (\forall y)Gy)$ on $I$.

By (9.), $d$ satisfies $(\exists x)(Fx \supset (\forall y)Gy)$ on $I$ iff for some $u \in UD$, $d[u/x]$ satisfies $(Fx \supset (\forall y)Gy)$ on $I$.

By (6.), for some $u \in UD$, $d[u/x]$ satisfies $(Fx \supset (\forall y)Gy)$ on $I$ iff for some $u \in UD$, either $d[u/x]$ doesn’t satisfy $Fx$ on $I$ or $d[u/x]$ satisfies $(\forall y)Gy$ on $I$.

Prove the RHS.
<b> So for arbitrary d, </b><i>d</i><sub>I,d[b/x]\right>(x) \not\in I(F).

So by (2.), d[b/x] doesn't satisfy Fx on I.

So for some u∈UD, d[u/x] doesn't satisfy Fx on I.

So for some u∈UD, either d[u/x] doesn't satisfy Fx on I or d[u/x] satisfies (∀y)Gy on I.

So for some u∈UD, d[u/x] satisfies (Fx ⊃ (∀y)Gy) on I.

So d satisfies (∃x)(Fx ⊃ (∀y)Gy) on I.

We picked an arbitrary d, so every d for I satisfies (∃x)(Fx ⊃ (∀y)Gy) on I.

So (∃x)(Fx ⊃ (∀y)Gy) is true on I.
Show quantificationally true:

\((\forall x)(Rxx \supset (\exists y)Rxy)\)

\((\forall x)(Rxx \supset (\exists y)Rxy)\) is q-true iff it's true on any I.

\((\forall x)(Rxx \supset (\exists y)Rxy)\) is true on any I iff for any I, every d for I satisfies \((\forall x)(Rxx \supset (\exists y)Rxy)\) on I.

Pick an arbitrary I and an arbitrary d.

d satisfies \((\forall x)(Rxx \supset (\exists y)Rxy)\) on I iff for any \(u \in \text{UD}\), \(d[u/x]\) satisfies \((Rxx \supset (\exists y)Rxy)\) on I.

(For any \(u \in \text{UD}\) \(d[u/x]\) satisfies \((Rxx \supset (\exists y)Rxy)\) on I) iff (for any \(u \in \text{UD}\), either \(d[u/x]\) doesn’t satisfy \(Rxx\) or \(d[u/x]\) satisfies \((\exists y)Rxy\) on I.

Prove that the right-hand side is true by reductio.
Suppose RHS is false. Then for some \( u \in UD \), \( d[u/x] \) satisfies \( Rxx \) on \( I \) and doesn’t satisfy \((\exists y)Rxy\) on \( I \).

Pick an arbitrary \( u \) such that \( d[u/x] \) satisfies \( Rxx \) on \( I \).

So by (2.), \( \langle \text{den}_{I,d[u/x][u/y]}(y), \text{den}_{I,d[u/x][u/y]}(x) \rangle \in I(R) \).

So \( \langle u,u \rangle \in I(R) \)

So for some \( v \in UD \), \( \langle \text{den}_{I,d[u/x][v/y]}(y), \text{den}_{I,d[u/x][v/y]}(x) \rangle \in I(R) \).

By (2.), for some \( v \in UD \), \( d[u/x][v/y] \) satisfies \( Rxy \) on \( I \).

By (9.), \( d[u/x] \) satisfies \((\exists y)Rxy\) on \( I \).

\( u \) was arbitrary, for it’s not the case that for some \( u \in UD \), \( d[u/x] \) satisfies \( Rxx \) on \( I \) and doesn’t satisfy \((\exists y)Rxy\) on \( I \).
Since it's not the case that for some \( u \in UD \), \( d[u/x] \) satisfies \( Rxx \) on \( I \) and doesn't satisfy \( (\exists y)Rxy \) on \( I \):

For any \( u \in UD \), either \( d[u/x] \) doesn't satisfy \( Rxx \) or \( d[u/x] \) satisfies \( (\exists y)Rxy \) on \( I \).

We had:

\[
(\text{For any } u \in UD, \text{ } d[u/x] \text{ satisfies } (Rxx \supset (\exists y)Rxy) \text{ on } I)
\]

iff (for any \( u \in UD \), either \( d[u/x] \) doesn't satisfy \( Rxx \) or \( d[u/x] \) satisfies \( (\exists y)Rxy \) on \( I \).

The RHS is true, so the LHS is too.

So for any \( u \in UD \), \( d[u/x] \) satisfies \( (Rxx \supset (\exists y)Rxy) \) on \( I \).

So by (8.), \( d \) satisfies \( (\forall x)(Rxx \supset (\exists y)Rxy) \) on \( I \).

\( d \) and \( I \) were arbitrary, so for any \( I \), for any \( d \), \( d \) satisfies \( (\forall x)(Rxx \supset (\exists y)Rxy) \).

So for any \( I \), \( (\forall x)(Rxx \supset (\exists y)Rxy) \) is true on \( I \).