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Quantum Information Science

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Problem Set #3 Solutions

Problems:

P1: (**Threshold estimates for Bacon-Shor code**)

P2: (**Quantum Fredkin gate via teleportation**)

(a) The Fredkin gate is a controlled-SWAP gate, so we can easily construct a Toffoli gate by applying a CNOT gate on the target qubits before and after the Fredkin gate. The Toffoli gate and Hadamard gate have been shown to be computationally universal in a paper by Shi (quant-ph/0205115). It turns out you don't need the state $|0\rangle + i|1\rangle$ after all(!).

(b) Let us begin with a tedious, but straightforward solution. We can represent the quantum Fredkin operator in the form

$$U_F = |0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J$$

where we have defined the operator J to be the swap operator on two qubits:

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, applying U_F to the operators in the set $\{IIX, IXI, IXX, XII, XIX, XXI, XXX\}$, gives:

- $g = IIX$:

$$\begin{aligned} U_g &= U_F(IIX)U_F \\ &= (|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J)(IIX)(|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J) \\ &= |0\rangle\langle 0| \otimes IX + |1\rangle\langle 1| \otimes J(IX)J \\ &= |0\rangle\langle 0| \otimes IX + |1\rangle\langle 1| \otimes XI \\ &= C_{1,2}C_{1,3}X_3 \\ &= C_{1,2}C_{1,3}H_3S_3^2H_3 \end{aligned}$$

where we have also defined the operator $C_{a,b}$ as a CNOT with control qubit a and target qubit b.

- $g = IXI$:

$$\begin{aligned} U_g &= |0\rangle\langle 0| \otimes XI + |1\rangle\langle 1| \otimes J(XI)J \\ &= |0\rangle\langle 0| \otimes XI + |1\rangle\langle 1| \otimes IX \\ &= C_{1,2}C_{1,3}H_2S_2^2H_2 \end{aligned}$$

- $g = IXX$:

$$\begin{aligned} U_g &= U_F(IXI)(IIX)U_F \\ &= U_F(IXI)U_FU_F(IIX)U_F \\ &= U_{IXI}U_{IIX} \\ &= H_2S_2^2H_2H_3S_3^2H_3 \end{aligned}$$

- $g = XII$:

$$\begin{aligned} U_g &= (|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J)(XII)(|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J) \\ &= |0\rangle\langle 1| \otimes J + |1\rangle\langle 0| \otimes J \\ &= X \otimes J \\ &= C_{2,3}C_{3,2}C_{2,3}H_1S_1^2H_1 \end{aligned} \tag{1}$$

- $g = XIX$:

$$\begin{aligned} U_g &= U_{XII}U_{IIX} \\ &= C_{2,3}C_{3,2}C_{2,3}H_1S_1^2H_1C_{1,2}C_{1,3}H_3S_3^2H_3 \end{aligned}$$

- $g = XXI$:

$$\begin{aligned} U_g &= U_{XII}U_{IXI} \\ &= C_{2,3}C_{3,2}C_{2,3}H_1S_1^2H_1C_{1,2}C_{1,3}H_2S_2^2H_2 \end{aligned}$$

- $g = XXX$:

$$\begin{aligned} U_g &= (|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J)(XXX)(|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J) \\ &= |0\rangle\langle 1| \otimes (XX)J + |1\rangle\langle 0| \otimes J(XX) \\ &= XXX \otimes IJ \\ &= C_{2,3}C_{3,2}C_{2,3}H_1S_1^2H_1H_2S_2^2H_2H_3S_3^2H_3 \end{aligned}$$

  It follows from the fact that $[J, XX] = 0$.

Similarly, for the set $\{IIZ, IZI, IZZ, ZII, ZIZ, ZZI, ZZZ\}$, we have:

- $g = IIZ$:

$$U_g = |0\rangle\langle 0| \otimes IZ + |1\rangle\langle 1| \otimes J(IZ)J$$

$$\begin{aligned} &= |0\rangle\langle 0| \otimes IZ + |1\rangle\langle 1| \otimes ZI \\ &= H_1 C_{2,1} C_{3,1} H_1 S_3^2 \end{aligned}$$

Note that $X_1\text{CPHASE}_{1,3}X_1 = \text{CPHASE}_{1,3}Z_3$ and $H_1 C_{2,1} H_1 = \text{CPHASE}_{1,2}$.

- $g = IZI$:

$$U_g = H_1 C_{2,1} C_{3,1} H_1 S_2^2$$

- $g = IZZ$:

$$U_g = S_2^2 S_3^2$$

- $g = ZII$:

$$\begin{aligned} U_g &= (|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J)(ZII)(|0\rangle\langle 0| \otimes II + |1\rangle\langle 1| \otimes J) \\ &= |0\rangle\langle 0| \otimes II - |1\rangle\langle 1| \otimes II \\ &= S_1^2 \end{aligned}$$

- $g = ZIZ$:

$$U_g = H_1 C_{2,1} C_{3,1} H_1 S_3^2 S_1^2$$

- $g = ZZI$:

$$U_g = H_1 C_{2,1} C_{3,1} H_1 S_2^2 S_1^2$$

- $g = ZZZ$:

$$U_g = S_1^2 S_2^2 S_3^2$$

Simpler solution. It is sufficient to consider the action of U on X_1, X_2, X_3, Z_1, Z_2, and Z_3 since $Ug_1g_2U^\dagger = Ug_1U^\dagger U g_2 U^\dagger$. By inspection, $UX_3U^\dagger = \Lambda_1(X_2)X_1\Lambda_1(X_3)X_1$ where $X_1\Lambda_1(X_3)X_1$ is a zero-controlled X_3 gate. Similarly, $UX_2U^\dagger = X_1\Lambda_1(X_2)X_1\Lambda_1(X_3)$, $UX_1U^\dagger = X_1 \otimes \text{SWP}_{2,3}$, $UZ_3U^\dagger = \Lambda_1(Z_2)X_1\Lambda_1(Z_3)X_1$, $UZ_2U^\dagger = X_1\Lambda_1(Z_2)X_1\Lambda_1(Z_3)$, and $UZ_1U^\dagger = Z_1$. It is possible to simplify products of these gates, i.e. $UX_2X_3U^\dagger = X_2X_3$.

(c) We start with the circuit in Figure 1 that uses three Bell states to teleport the qubits and applies the Fredkin gate on them. Now, the only thing that we have to do is to commute the classically controlled correction gates to the other side of the Fredkin gate as illustrated in Figure 2. The resulting circuit uses $|\chi\rangle$ and a few classically controlled operations to give a realization for U_F.

Alternate solution. Another approach to this problem is to use the "X" and "Z" one-bit teleportation circuits presented in lecture. In particular, the Fredkin gate commutes with ZZZ and with ZXX, so there are two ways of constructing the circuit. Let F denote the Fredkin gate. If we use XXX

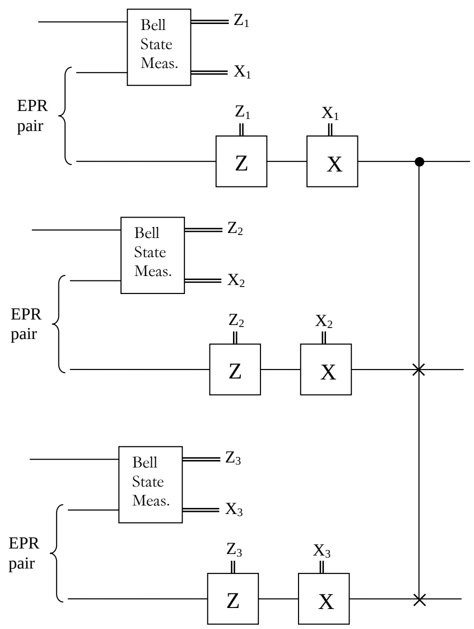


Figure 1: Teleportation of three qubits followed by Fredkin gate.

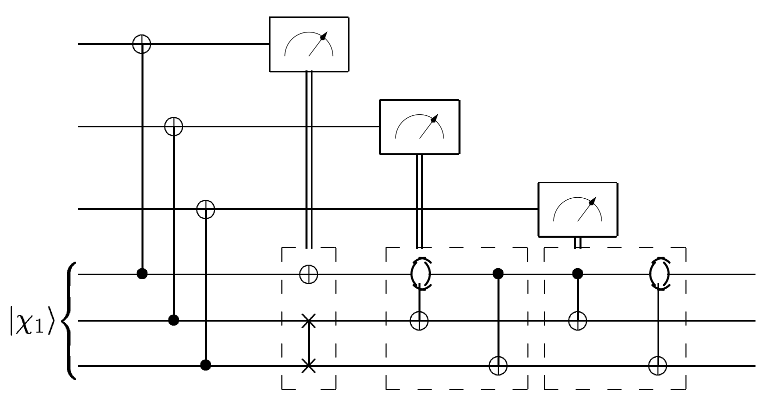
teleportation, the state we need is the equal superposition

$$|\chi_1\rangle = FH_1H_2H_3|000\rangle = \frac{1}{\sqrt{8}} \sum_{z_1,z_2,z_3=0}^{1} |z_1z_2z_3\rangle \,. \tag{2}$$

If we use XZZ teleportation, the state we need is

$$|\chi_2\rangle = FH_1|000\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \,. \tag{3}$$

P3: (**1.**) Using XXX teleportation, we find the following circuit:



If we use XZZ teleporation instead, we find the following circuit:

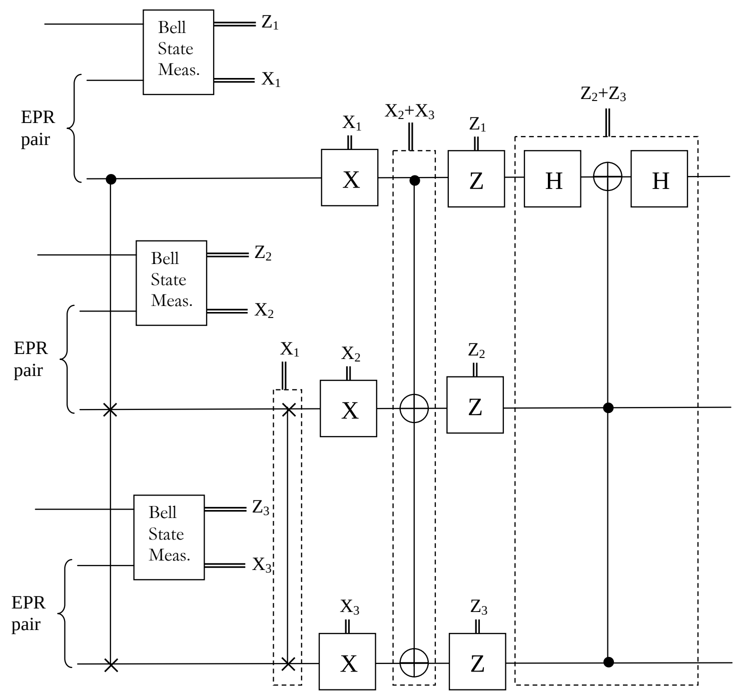
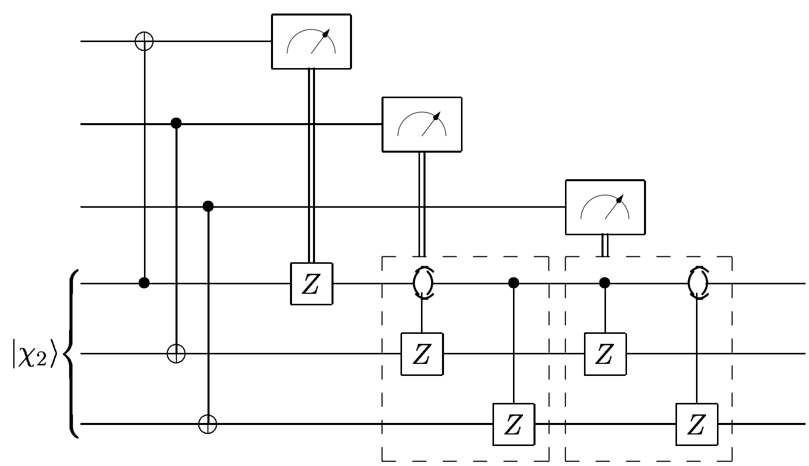


Figure 2: Quantum circuit for teleportation of Fredkin gate.



P4: (**Cluster model implementation of quantum Fourier transform**)

(a) In the quantum Fourier transform circuit, there are 3 controlled R_z gates (S and T), 3 H gates and 1 swap gate which is equivalent to 3 CNOTs. Moreover, in order to implement a controlled R_z gate, we need 2 CNOTs and 2 R_z gates. On the other hand, cluster state realization of any R_z or H gate requires adding one new qubits to the cluster while for CNOTs 2 new qubits are needed. Putting all these numbers together, we will aggregately need $3 \times (2 + 2 \times 2) + 3 \times 2 + 3 + 3 = 30$ qubits in the cluster state. Note that we have also added 3 qubits for the output of the circuit that is not measured in the process.

(b) If we find two operators A and B such that $AB = I$ and $AXBX = R_z(\theta)$, we can obtain a controlled $R_z(\theta)$ using the circuit in Figure 3.

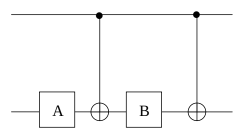


Figure 3: Quantum circuit for decomposition controlled rotation.

Let $A = R_z(\alpha)$ and $B = R_z(\beta)$. Hence,

$$R_z(\theta) = AXBX = R_z(\alpha)XR_z(\beta)X = R_z(\alpha)R_z(-\beta) = R_z(\alpha - \beta) \Longrightarrow \theta = \alpha - \beta$$

$$I = AB = R_z(\alpha)R_z(\beta) = R_z(\alpha + \beta) \Longrightarrow 2\pi = \alpha + \beta$$

and by choosing $\alpha = 5\pi/4$, $\beta = 3\pi/4$ or $\alpha = 9\pi/8$, $\beta = 7\pi/8$ we can implement controlled-S or controlled-T out of CNOT and R_z operations that have known implementations in cluster state QC.

(c) This circuit is not a very efficient cluster implementation, but is perhaps the most straightforward one.

Alternative solutions

(a)

There are cluster state implementations of $R_x(\theta)$, $R_z(\theta)$, and CNOT that require 2, 2, and 4 intermediate qubits, respectively, shown in Figure 5. The notation is taken from quant-ph/0404082.

Any single qubit unitary can be Euler decomposed as $e^{i\phi}R_z(\theta_1)R_x(\theta_2)R_z(\theta_3)$. Also, Figure 4.6 in N&C also shows how to implement an arbitrary controlled-U gate using 2 CNOTs and 3 single qubit gates.

The 3-qubit quantum fourier transform has 3 single qubits gates and three controlled unitary gates that can be implemented using 24 Z rotations, 12 X rotations, and 6 CNOTs. Therefore, 96 qubits and 3 output qubits are sufficient to simulate the 3-qubit quantum fourier transform using a cluster state. We will do a little better in part (b).

(b)

Since $R_z(\theta/2)R_z(-\theta/2) = I$ and $(R_z(\theta/2)X)(R_z(-\theta/2)X) = R_z(\theta)$, $\Lambda(R_z(\theta)) = R_z(\theta/2)\Lambda(X)R_z(-\theta/2)\Lambda(X)$. The Euler decompositions for the remaining gates in the 3-qubit quantum fourier transform are $S = e^{-i\pi/4}R_z(\pi/2)$, $T = e^{-i\pi/8}R_z(\pi/4)$ and $H = e^{i\pi/2}R_z(\pi/2)R_x(\pi/2)R_z(\pi/2)$. The 3-qubit fourier transform circuit is reexpressed as single qubit rotations in Figure 6. Figure 7 shows a cluster state and a measurement procedure for the 3-qubit quantum fourier transform that uses 67 qubits.

(c)

This is not optimal. We can remove all pairs of adjacent X measurements, since these are "wires". Furthermore, Hadamard can be implemented by a single X measurement.

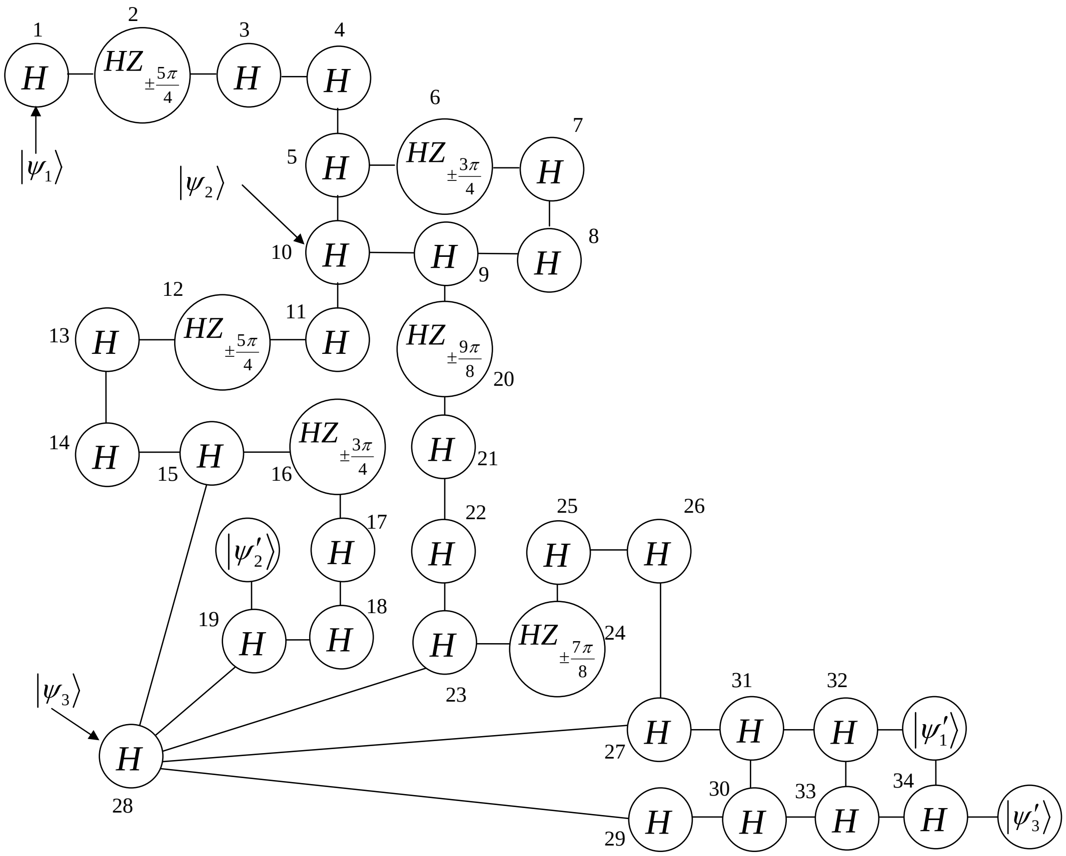


Figure 4: Cluster circuit implementation of three-qubit QFT.

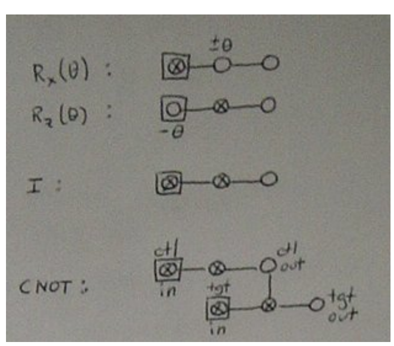


Figure 5: Measurement patterns for a universal set of gates

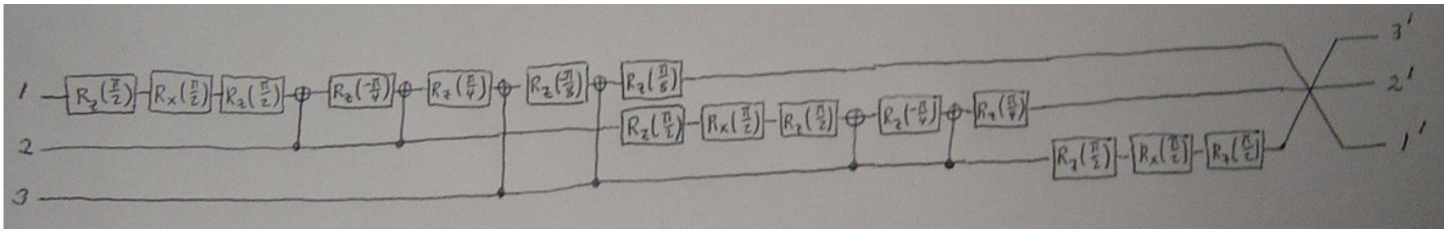


Figure 6: Further decomposition of 3-qubit QFT

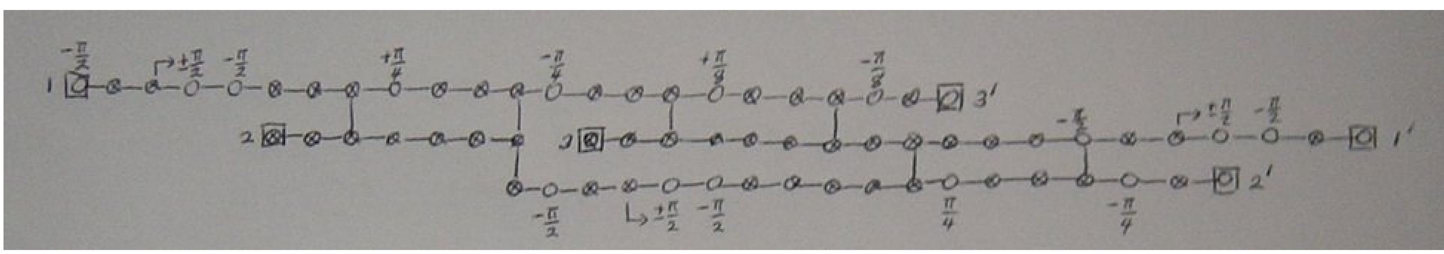


Figure 7: Cluster state and measurement pattern for 3-qubit QFT