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Quantum Information Science

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Problem Set #4 Solutions

Problems:

P1: (**Quantum factoring as a feedback process**)

See attached document.

P2: (**Measures of pure state entanglement**) Begin by recalling the proof of the Schmidt decomposition. Let $|\psi\rangle = \sum_{ij} c_{ij}|i\rangle|j\rangle = (C \otimes I)\sum_i |i\rangle|i\rangle$ where $C := \sum_{ij} |i\rangle\langle j|$. By the singular value decomposition, $C = A\Lambda B$, where A, B are unitary, $\Lambda = \sum_k \lambda_k|k\rangle\langle k|$ with $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$ and Λ is uniquely determined. (If there are degenerate eigenvalues then A and B are not uniquely determined.) Also $1 = \sum_{ij}|c_{ij}|^2 = \mathrm{tr}C^\dagger C = \mathrm{tr}\Lambda^\dagger\Lambda = \sum_k \lambda_k^2$. Thus $|\psi\rangle = (A\Lambda B \otimes I)\sum_i |i\rangle|i\rangle = (A \otimes B^T)(\Lambda \otimes I)\sum_i |i\rangle|i\rangle$, since $(B \otimes I)\sum_i |i\rangle|i\rangle = (I \otimes B^T)\sum_i |i\rangle|i\rangle$. Define $|k_A\rangle = A|i\rangle$ and $|k_B\rangle = B|i\rangle$ and we have

$$|\psi\rangle = \sum_k \lambda_k |k_A\rangle|k_B\rangle$$

(a) If $\mathrm{Sch}(|\psi\rangle) = 1$ then $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$ follows from the definition. The converse is a special case of the agument below.

Note that if a bipartite state $|\psi\rangle$ can be expressed as any state of the form $|\psi\rangle = \sum_k |\phi_k\rangle|k_B\rangle$, where $|k_B\rangle$ are orthonormal states of B and $|\phi_k\rangle$ are arbitrary (possibly un-normalized) states of A, then the number of terms in the sum is at least as great as the Schmidt number of $|\psi\rangle$.

Since $\mathrm{rank}\ (\mathrm{A} + \mathrm{B}) \leq \mathrm{rank}\ \mathrm{A} + \mathrm{rank}\ \mathrm{B}$,

$$\mathrm{Sch}(\psi) = \mathrm{rank}\ \mathrm{tr_B}|\psi\rangle\langle\psi| = \sum_{\mathrm{k}} |\phi_k\rangle\langle\phi_k| \leq \sum_{\mathrm{k}} 1,$$

where the final expression is simply the number of terms in the original sum.

This fact also holds for decompositions in which Bob's states are not orthonormal either. If $|\psi\rangle = \sum_k |\phi_k\rangle|\varphi_k\rangle$, then $\rho = \mathrm{tr}_B|\psi\rangle\langle\psi| = \sum_{k,l} |\phi_k\rangle\langle\phi_l|\langle\varphi_k||\varphi_l\rangle$ and for any $|v\rangle$, $\rho|v\rangle \in \mathrm{span}\{|\phi_k\rangle\}$, which has dimension at most equal to the number of terms in the original sum.

(b) Suppose Alice applies U and Bob applies V. Then $(U \otimes V)|\psi\rangle = \sum_k \lambda_k U|k_A\rangle \otimes V|k_B\rangle$ and by the uniqueness of the Schmidt decomposition, the Schmidt number is unchanged. Classical communication has no effect unless Alice or Bob performs a measurement since any classical message must be uncorrelated with the state.

However, we can also show that Schmidt number is *nonincreasing* under local measurement and classical communication, though not constant. Suppose Alice performs a local measurement $\{M_j\}$

and transmits the outcome j to Bob. The resulting state is $(M_j \otimes I)|\psi\rangle = \sum_k \lambda_k (M_j|k_A\rangle) \otimes |k_B\rangle$ and by part (d), this has Schmidt number no greater than the number of terms in the sum, which is $\mathrm{Sch}(|\psi\rangle)$.

(c) The Schmidt numbers for $|\phi_1\rangle$ and $|\phi_2\rangle = |+\rangle|+\rangle$ are 3 and 1, respectively. Also,

$$\begin{aligned} |\phi_3\rangle &= \frac{|0\rangle|+\rangle + |1\rangle|-\rangle}{2}, && \text{Sch.N.} = 1, \\ |\phi_4\rangle &: \rho_A = \mathrm{tr}_B|\phi_4\rangle\langle\phi_4| = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, && \text{Sch.N.} = 2. \end{aligned}$$

P3: (**Quantum search by continuous-time simulation**) //

(a)

$$\begin{aligned} U(\Delta t) &= U_\psi(\Delta t) U_x(\Delta t) && (1)\\ &= e^{-i|\psi\rangle\langle\psi|\Delta t} e^{-i|\psi\rangle\langle\psi|\Delta t} && (2)\\ &= e^{-i\frac{I+\vec{\psi}\cdot\vec{\sigma}}{2}\Delta t} e^{-i\frac{I+\vec{z}\cdot\vec{\sigma}}{2}\Delta t} && (3)\\ &= \left(\cos\left(\frac{\Delta t}{2}\right) - i\sin\left(\frac{\Delta t}{2}\right)(I + \vec{\psi}\cdot\vec{\sigma})\right)\left(\cos\left(\frac{\Delta t}{2}\right) - i\sin\left(\frac{\Delta t}{2}\right)(I + \vec{z}\cdot\vec{\sigma})\right) && (4)\\ &= \cos^2\left(\frac{\Delta t}{2}\right) - i\cos\left(\frac{\Delta t}{2}\right)\sin\left(\frac{\Delta t}{2}\right)(I + \vec{z}\cdot\vec{\sigma}) && (5)\\ &\quad -i\cos\left(\frac{\Delta t}{2}\right)\sin\left(\frac{\Delta t}{2}\right)(I + \vec{\psi}\cdot\vec{\sigma}) - \sin^2\left(\frac{\Delta t}{2}\right)(I + \vec{z}\cdot\vec{\sigma})(I + \vec{\psi}\cdot\vec{\sigma}). && (6) \end{aligned}$$

Using the identity

$$(\vec{\psi}\cdot\vec{\sigma})(\vec{z}\cdot\vec{\sigma}) = \vec{\psi}\cdot\vec{z} + (\vec{\psi}\times\vec{z})\cdot\sigma\,, \tag{7}$$

we have that

$$\begin{aligned} U(\Delta t) = &\left(\cos^2\left(\frac{\Delta t}{2}\right) - \sin^2\left(\frac{\Delta t}{2}\right)\vec{\psi}\cdot\hat{z}\right) I \\ &-2i\sin\left(\frac{\Delta t}{2}\right)\left(\cos\left(\frac{\Delta t}{2}\right)\frac{\vec{\psi}+\hat{z}}{2} + \sin\left(\frac{\Delta t}{2}\right)\frac{\vec{\psi}\times\hat{z}}{2}\right)\cdot\vec{\sigma}\,, \end{aligned} \tag{8}$$

(b) Substituting for $\Delta t = \pi$ in $e^{-i\frac{I+\psi\cdot\sigma}{2}}$ and $e^{-i\frac{I+\vec{z}\cdot\vec{\sigma}}{2}}$, we get

$$U_\psi(\pi) = e^{-i|\psi\rangle\langle\psi|\pi} = I - 2|\psi\rangle\langle\psi| \tag{9}$$

and

$$U_x(\pi) = e^{-i|x\rangle\langle x|\pi} = I - 2|x\rangle\langle x|\,, \tag{10}$$

which are the same as the Grover search iteration steps up to a global phase factor.

(c) We want to get to $\hat{z}$ from $\vec{\psi}$ with K times rotating around the axis $\vec{r}$ (Eq.(6.26), Nielsen & Chuang) each time the angle θ. K is required to be of the order $\mathcal{O}(\sqrt{N})$. We approximate the total rotation angle by the angle between $\hat{z}$ and $\vec{\psi}$, that is, $\cos^{-1}(\hat{z}\cdot\vec{\psi})$.

$$\begin{aligned}
K(\sqrt{N})\theta &= \cos^{-1}\left(\hat{z}\cdot\vec{\psi}\right) \\
&= \cos^{-1}\left(\alpha^2-\beta^2\right) \\
&= \cos^{-1}\left(\frac{2}{N}-1\right) \\
&= \pi-\cos^{-1}\left(1-\frac{2}{N}\right) \\
&\Rightarrow \theta = \frac{\pi-\cos^{-1}\left(1-\frac{2}{N}\right)}{K(\sqrt{N})}\,. \qquad (11)
\end{aligned}$$

Now, substituting for θ in (Eq.(6.28), Nielsen & Chuang), we get

$$\sin^2\left(\frac{\Delta t}{2}\right) = \frac{N}{2}\left[1-\cos\left(\frac{\pi-\cos^{-1}\left(1-\frac{2}{N}\right)}{K(\sqrt{N})}\right)\right]. \qquad (12)$$

The smallest integer for K that makes the right hand side less that 1, gives the appropriate choice for Δt.

P4: (**Typical sequences (computational)**)

(a)

$$\begin{aligned}
H(X) &= H(X_1) && (13)\\
&= -0.8\log 0.8 - 0.1\log 0.1 - 0.1\log 0.1 && (14)\\
&\approx 0.922\,. && (15)
\end{aligned}$$

(b) In general, a sequence is ϵ-typical when

$$2^{-n(H(X)+\epsilon)} \le p(x_1,...,x_n) \le 2^{-n(H(X)-\epsilon)}\,. \qquad (16)$$

So, in this case, a sequence is ϵ-typical when

$$2^{-n(0.922+\epsilon)} \le 0.8^{n_a}0.1^{n-n_a} \le 2^{-n(0.922-\epsilon)}\,. \qquad (17)$$

(c)

$$2^{-100(1.022)} \le 0.8^{n_a}0.1^{100-n_a} \le 2^{-100(0.822)}\,, \qquad (18)$$

so

$$77 \le n_a \le 84\,. \qquad (19)$$

Thus,

$$\Pr(A_0^{(100)}.1) = \sum_{i=77}^{84} Pr(n_a = i) \tag{20}$$

$$= \sum_{i=77}^{84} 2^{n-i} \binom{100}{i} 0.8^i 0.1^{n-i} \tag{21}$$

$$= 0.682\,. \tag{22}$$

(d)

$$|A_{0.1}^{(100)}| = \sum_{i=77}^{84} 2^{n-i} \binom{100}{i}\,, \tag{23}$$

which requires 98 bits to represent; this is very close to the value of $nH(X_1) = 92$.