3.032 Mechanical Behavior of Materials
Fall 2007

STRESS AND STRAIN TRANSFORMATIONS:
Finding stress on a material plane that differs from the one on which stress is known...
or "Why it’s easier to remember Mohr’s circle"
Note: Derived in class on Wednesday 09.19.07.

Force balance for stress over a face inclined an angle \( \theta \) with respect to the original (x, y) axes give:

\[
\sigma'_{x,x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

(1)

\[
\sigma'_{y,y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta
\]

(2)

\[
\tau'_{x,y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

(3)

Taking the derivative of Eq. (1) with respect to \( \theta \) to obtain the orientation of maximum normal stress gives:

\[
tan 2\theta_{\text{normal stress, max}} = \frac{\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}
\]

(4)

and substituting the corresponding \( \sin 2\theta \) and \( \cos 2\theta \) expressions into Eqs. (1 - 2) to obtain the maximum normal stresses in this 1-2 plane gives:

\[
\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}
\]

(5)

where, by convention, \( \sigma_1 \geq \sigma_2 \).

Taking the derivative of Eq. (2) and going through the same process to obtain the orientation and magnitude of the maximum shear stresses gives:

\[
tan 2\theta_{\text{shear stress, max}} = \frac{\sigma_{xx} - \sigma_{yy}}{\tau_{xy}}
\]

(6)

and

\[
\tau_{\text{max, in-plane}} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}
\]

(7)

Note that the equations for coordinate transformations of strain (strain transformation equations) are completely analogous. For example,

\[
\epsilon'_{x,x'} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} + \frac{\epsilon_{xx} - \epsilon_{yy}}{2} \cos 2\theta + \epsilon_{xy} \sin 2\theta
\]

(8)
but the only thing to note is that this $\epsilon_{xy}$ is equal to half the engineering shear strain, $\gamma_{xy}/2$. In other words, if you are given a state of engineering strain for a material body, you have to multiply the engineering shear strain components by 2 before using these equations to find the full strain state on some other plane inclined an angle $\theta$.

As you will see in the next class, a very smart engineer named Otto Mohr figured out how to represent these equations in the shape of a circle, so that one can quickly and graphically locate the orientations and magnitudes of maximum normal and shear stresses / strains!