Matrix and Index Notation

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A vector can be described by listing its components along the $xyz$ cartesian axes; for instance the displacement vector $\mathbf{u}$ can be denoted as $u_x, u_y, u_z$, using letter subscripts to indicate the individual components. The subscripts can employ numerical indices as well, with 1, 2, and 3 indicating the $x$, $y$, and $z$ directions; the displacement vector can therefore be written equivalently as $u_1, u_2, u_3$.

A common and useful shorthand is simply to write the displacement vector as $u_i$, where the $i$ subscript is an index that is assumed to range over $1, 2, 3$ (or simply 1 and 2 if the problem is a two-dimensional one). This is called the range convention for index notation. Using the range convention, the vector equation $u_i = a$ implies three separate scalar equations:

\[
\begin{align*}
  u_1 &= a \\
  u_2 &= a \\
  u_3 &= a
\end{align*}
\]

We will often find it convenient to denote a vector by listing its components in a vertical list enclosed in braces, and this form will help us keep track of matrix-vector multiplications a bit more easily. We therefore have the following equivalent forms of vector notation:

\[
\mathbf{u} = u_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}
\]

Second-rank quantities such as stress, strain, moment of inertia, and curvature can be denoted as $3 \times 3$ matrix arrays; for instance the stress can be written using numerical indices as

\[
|\sigma| = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}
\]

Here the first subscript index denotes the row and the second the column. The indices also have a physical meaning, for instance $\sigma_{23}$ indicates the stress on the 2 face (the plane whose normal is in the 2, or $y$, direction) and acting in the 3, or $z$, direction. To help distinguish them, we’ll use brackets for second-rank tensors and braces for vectors.

Using the range convention for index notation, the stress can also be written as $\sigma_{ij}$, where both the $i$ and the $j$ range from 1 to 3; this gives the nine components listed explicitly above.
(Since the stress matrix is symmetric, i.e. $\sigma_{ij} = \sigma_{ji}$, only six of these nine components are independent.)

A subscript that is repeated in a given term is understood to imply summation over the range of the repeated subscript; this is the summation convention for index notation. For instance, to indicate the sum of the diagonal elements of the stress matrix we can write:

$$\sigma_{kk} = \sum_{k=1}^{3} \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

The multiplication rule for matrices can be stated formally by taking $A = (a_{ij})$ to be an $(M \times N)$ matrix and $B = (b_{ij})$ to be an $(R \times P)$ matrix. The matrix product $AB$ is defined only when $R = N$, and is the $(M \times P)$ matrix $C = (c_{ij})$ given by

$$c_{ij} = \sum_{k=1}^{N} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iN}b_{Nj}$$

Using the summation convention, this can be written simply

$$c_{ij} = a_{ik}b_{kj}$$

where the summation is understood to be over the repeated index $k$. In the case of a $3 \times 3$ matrix multiplying a $3 \times 1$ column vector we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{bmatrix} = c_{ij}b_j$$

The comma convention uses a subscript comma to imply differentiation with respect to the variable following, so $f_{,2} = \partial f / \partial y$ and $u_{i,j} = \partial u_i / \partial x_j$. For instance, the expression $\sigma_{ij,j} = 0$ uses all of the three previously defined index conventions: range on $i$, sum on $j$, and differentiate:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

The Kronecker delta is a useful entity is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

This is the index form of the unit matrix $I$:

$$\delta_{ij} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

So, for instance
\[ \sigma_{kk} \delta_{ij} = \begin{bmatrix} \sigma_{kk} & 0 & 0 \\ 0 & \sigma_{kk} & 0 \\ 0 & 0 & \sigma_{kk} \end{bmatrix} \]

where \( \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} \).