Honeycombs - In-plane behaviour

- Prismatic cells
- Polymer, metal, ceramic honeycombs widely available
- Used for sandwich structure cores, energy absorption, carriers for catalysts
- Some natural materials (e.g. wood, cork) can be idealized as honeycombs
- Mechanisms of deformation & failure in hexagonal honeycombs parallel those in foams
  - Simpler geometry (unit cell) - easier to analyze
- Mechanisms of deformation in triangular honeycombs parallels those in 3D trusses (lattice materials)

Stress - Strain Curves + Deformation behaviour: In-Plane

- Compression
  - 3 regimes - linear elastic - bending
    - Stress plateau - buckling
      - Yielding
      - Brittle crushing
    - densification - cell walls touch
  - Increasing $t/l = 0$ $E* \uparrow$ $\sigma* \uparrow$ $\epsilon_d \downarrow$
Honeycomb Geometry

Deformation mechanisms

Bending

X₁ Loading

Bending X₂ Loading

Bending Shear

Buckling

Plastic collapse in an aluminum honeycomb

Stress-Strain Curve

tension

- linear elastic - bending
- stress plateau - exists only if cell walls yield
  - no buckling in tension
  - brittle honeycombs fracture in tension

Variables affecting honeycomb properties

- relative density \( \rho^* = \frac{t/l}{h \sqrt{h^2 + 2}} \)
  \[ \frac{2 \cos \theta (h \sin \theta)}{2 \cos \theta (h \sin \theta)} = \frac{2}{\sqrt{3}} \]
  regular hexagons

- solid cell wall properties: \( \rho_s, E_s, C_{ys}, C_{fs} \)
- cell geometry: \( h, l, \theta \)
In-plane properties
assumptions:
- the small ($\rho^*/\rho_s$ small) - neglect axial + shear contribution to def
- deformations small - neglect changes in geometry
- cell well - linear elastic, isotropic

symmetry
- honeycombs are orthotropic - rotate 180° about each of three mutually perpendicular axes & structure is the same

Linear elastic deformation

\[
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
G_{12} \\
G_{13} \\
G_{23}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
\nu_{12} & \nu_{23} & \frac{1}{G_{23}} & 0 & 0 & 0 \\
\nu_{13} & \nu_{32} & \frac{1}{G_{32}} & 0 & 0 & 0 \\
\nu_{23} & \frac{1}{G_{32}} & \frac{1}{G_{13}} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_{4} \\
\sigma_{5} \\
\sigma_{6}
\end{bmatrix}
\]
Matrix notation:

\[ \begin{align*}
E_1 &= E_{11} & \sigma_1 &= \sigma_{11} & \sigma_4 &= \sigma_{23} \\
E_2 &= E_{22} & \sigma_2 &= \sigma_{22} & \sigma_5 &= \sigma_{13} \\
E_3 &= E_{33} & \sigma_3 &= \sigma_{33} & \sigma_6 &= \sigma_{12}
\end{align*} \]

In-plane \((x_1-x_2)\): 4 independent elastic constants:

\[ E_1, E_2, \nu_{12}, G_{12} \]

and compliance matrix symmetric: \( -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} \) (reciprocal relation)

\[ \text{Notation for Poisson's ratio: } \nu_{ij} = -\frac{E_j}{E_i} \]

Young's modulus in \(x_1\) direction:

\[ \sigma_1 = \frac{P}{(h + l \sin \theta)} b \]

\[ \epsilon_1 = \frac{\delta \sin \theta}{l \cos \theta} \]

Unit cell in \(x_1\) direction: \(2l \cos \theta\)
In-Plane Deformation: Linear Elasticity

Figure removed due to copyright restrictions. See Figure 5: L. J. Gibson, M. F. Ashby, et al. "The Mechanics of Two-Dimensional Cellular Materials."
M diagram: 2 cantilevers of length $\ell/2$

\[
\delta = 2 \cdot \frac{P \sin \theta (\ell/2)^3}{3 E_s I} = \frac{2 P \ell^2 \sin \theta}{24 E_s I}
\]

\[
\delta = \frac{P \ell^3 \sin \theta}{12 E_s I}, \quad I = \frac{bt^3}{12}
\]

Combining:

\[
E_1^* = \frac{E_s (t/l)^3}{(h + l \sin \theta) b} \cdot \frac{l \cos \theta}{\delta \sin \theta}
\]

\[
= \frac{P}{(h + l \sin \theta)b} \cdot \frac{l \cos \theta}{P \ell^3 \sin \theta} \cdot \frac{1/2 E_s b t^3}{1/2}
\]

\[
E_1^* = \frac{E_s (t/l)^3}{(h/l \cos \theta + \sin \theta) \sin^2 \theta} \cos \theta = \frac{4 (t/l)^3 E_s}{h/l \cos \theta + \sin \theta} \text{ regular hexagons}
\]

\[h/l = 1, \theta = 30^\circ\]
Poisson's ratio for loading in $x_1$ direction

$\nu_{12}^* = -\frac{E_2}{E_1}$

$\varepsilon_1 = -\frac{\delta \sin \theta}{l \cos \theta}$ (shortens)

$\varepsilon_2 = \frac{\delta \cos \theta}{h + ls \sin \theta}$ (lengthens)

$\nu_{12}^* = -\frac{\delta \cos \theta}{h + ls \sin \theta} \left( \frac{l \cos \theta}{\delta \sin \theta} \right) = \frac{\cos^2 \theta}{(h \frac{1}{2} + s \sin \theta) \sin \theta}$

- $\nu_{12}^*$ depends only on cell geometry $(h, l, \theta)$, not on $E_1, E_2$
- regular hexagonal cells: $\nu_{12}^* = 1$
- $\nu$ can be negative for $\theta < 0$
  - eg. $h_1 = 2, \theta = -30^\circ$: $\nu_{12}^* = \frac{3/4}{(3/2)(-1/2)} = -1$

$E_2^* \nu_{12}^* G_{12}^*$

- can be found in similar way; results in book.
Compressive strength (plateau stress)

- cell collapse by:
  - (1) elastic buckling
  - (2) plastic yielding
  - (3) brittle crushing

\[ \sigma - \varepsilon \]

- buckling of vertical struts throughout honeycomb
- localization of yield as deformation progresses, propagation of failure band
- peaks and valleys correspond to fracture of individual cell walls

Plateau stress: elastic buckling, \( \sigma_{el} \)

- Elastic honeycombs - cell collapse by elastic buckling of walls of length \( h \) when loaded in \( x_2 \) direction
- No buckling for \( \sigma_1 \); bending of inclined walls goes to densification.

Euler buckling load

\[ P_{el} = \frac{n^2 \pi^2 E_s I}{h^2} \]

- \( n = \) end constraint factor
- \( \text{pm-pm} \) \( n = 1 \)
- \( \text{fixed-fixed} \) \( n = 2 \)
Elastic Buckling

Figure removed due to copyright restrictions. See Figure 7: L. J. Gibson, M. F. Ashby, et al. "The Mechanics of Two-Dimensional Cellular Materials."
here, constraint n depends on stiffness of adjacent inclined members

- can find by elastic line analysis (see appendix if interested)
- rotational stiffness at ends of column, h, matched to rotational stiffness of inclined members

\[ \frac{h}{l} = 1 \quad 1.5 \quad 2 \]

\[ n = 0.686 \quad 0.760 \quad 0.860 \]

and \( \left( \sigma_{e1}^* \right)_2 = \frac{Pcr}{2 \ell \cos \theta b} = \frac{n^2 \pi^2 E_s}{h^2 2 \ell \cos \theta b} \frac{b t^3}{12} \)

\[
\left( \sigma_{e1}^* \right)_2 = \frac{n^2 \pi^2 E_s}{24} \frac{(t/l)^3}{(h/l)^2 \cos \theta}
\]

regular hexagons: \( \left( \sigma_{e1}^* \right)_2 = 0.22 E_s (t/l)^3 \)

\[ \text{since} \quad E_s^* = \frac{4}{13} E_s (t/l)^3 = 2.31 E_s (t/l)^3 \]

strain at buckling \( \left( E_{e1}^* \right)_2 = 0.10 \), for regular hexagons independent of \( E_s, t/l \)
Plateau stress: plastic yielding, $\sigma^p$:

- failure by yielding in cell walls
- yield strength of cell walls = $\sigma_{ys}$
- plastic hinge forms when cross-section fully yields
- beam theory - linear elastic $\sigma = \frac{My}{I}$

\[ \sigma \]

\[ t \quad \text{neutral axis, N.A.} \]

- Once stress at outer fiber = $\sigma_{ys}$, yielding begins & then progresses through the section, as the load increases

as $P \uparrow$

- When section fully yielded (right fig), form plastic "hinge"
- section rotates, like a pin
Plastic Collapse

Figure removed due to copyright restrictions. See Figure 8: L. J. Gibson, M. F. Ashby, et al. "The Mechanics of Two-Dimensional Cellular Materials."
- Moment at formation of plastic hinge (plastic moment, $M_p$):
  \[ M_p = \left( \sigma_{ys} b t \right) \left( \frac{t}{2} \right) = \frac{\sigma_{ys} b t^2}{4} \]

- Applied moment, from applied stress
  \[ 2 M_{applied} - Pl \sin \theta = 0 \]
  \[ M_{applied} = \frac{Pl \sin \theta}{2} \]

- Plastic collapse of honeycomb at $(\sigma_{pl}^x)_1$, when $M_{applied} = M_p$
  \[ (\sigma_{pl}^x)_1 = \frac{\sigma_{ys} (\frac{t}{2})^2}{2 \left( \frac{h}{l} + \sin \theta \right) \sin \theta} \]

  \[
  (\sigma_{pl}^x)_1 = \sigma_{ys} \left( \frac{t}{2} \right)^2 \frac{1}{2 \left( \frac{h}{l} + \sin \theta \right) \sin \theta}
  \]

  Regular hexagons:
  \[ (\sigma_{pl}^x)_1 = \frac{2}{3} \sigma_{ys} \left( \frac{t}{2} \right)^2 \]

  Similarly,
  \[ (\sigma_{pl}^x)_2 = \sigma_{ys} \left( \frac{t}{2} \right)^2 \frac{1}{2 \cos^2 \theta} \]

- Can do similar calculations for $(\sigma_{pl}^x)_3$.
for thin-walled honeycombs, elastic buckling can precede plastic collapse (for $E_s$)

\[ \text{elastic buckling stress} = \text{plastic collapse stress} \quad (\sigma_{	ext{el}}^*)^2 = (\sigma_{	ext{pl}}^*)^2 \]

\[ \frac{n^2 \pi^2 E_s (t/l)^3}{12 (t/l)^2 \cos \theta} \frac{(t/l)^2}{2 \cos^2 \theta} = \frac{\sigma_{\text{YS}}}{E_s} \]

\[ (t/l)_{\text{critical}} = \frac{12 (t/l)^2}{n^2 \pi^2 \cos \theta} \left( \frac{\sigma_{\text{YS}}}{E_s} \right) \]

regular hexagons: \( (t/l)_{\text{critical}} = 3 \frac{\sigma_{\text{YS}}}{E_s} \)

\[ \text{e.g. metals } \sigma_{\text{YS}}/E_s \sim 0.002 \quad (t/l)_{\text{crit}} \sim 0.6\% \]

most metal honeycombs denser than this

polymers \( \sigma_{\text{YS}}/E_s \sim 3-5\% \quad (t/l)_{\text{crit}} \sim 10-15\% \)

low density polymers with yield point may buckle before yield.
Plateau stress: brittle crushing, \( (\sigma^*_{cr})_1 \)
- ceramic honeycombs - fail in brittle manner
- cell wall bending - stress reaches modulus of rupture - wall fracture loading in \( x_1 \) direction:

\[
P = \sigma_1 (h + l\sin\theta)b \\
\sigma_{fs} = \text{modulus of rupture of cell wall}
\]

\[
M_{\text{max applied}} = \frac{P b l \sin\theta}{2} = \frac{\sigma_1 (h + l\sin\theta) b l \sin\theta}{2}
\]

Moment at fracture, \( M_f \):

\[
M_f = \left(\frac{1}{2} \sigma_{fs} b t \right) \left(\frac{2t}{3}\right) = \frac{\sigma_{fs} b t^2}{6}
\]

\[
(\sigma^*_{cr})_1 = \sigma_{fs} \left(\frac{t}{l}\right)^2 \left(\frac{1}{\frac{1}{3}(\frac{h}{l} + \sin\theta)}\right) \frac{1}{\sin\theta}
\]

regular hexagons: \( (\sigma^*_{cr})_1 = \frac{4}{9} \sigma_{fs} \left(\frac{t}{l}\right)^2 \)
Brittle Crushing

Tension

- no elastic buckling
- plastic plateau stress approx same in tension + compression (small geometric difference due to deformation)
- Brittle honeycombs: fast fracture

Fracture toughness

- crack length large relative to cell size (continuum assumption)
- axial forces can be neglected
- cell wall material has constant modulus of rupture, $\sigma_f$

Continuum: Crack of length $2c$. In a linear elastic solid material, normal to a remote tensile stress $\sigma_1$, creates a local stress field at the crack tip

$\sigma_{local} = \sigma_2 = \frac{\sigma_1 \sqrt{\pi c}}{\sqrt{2\pi r}}$
Fracture Toughness

honeycomb: cell walls bend - fail when applied moment = fracture moment

Mapp a Pl on wall A \[ M_f \propto \sigma_{fs} b t^2 \]

Mapp a Pl a \[ \sigma_d l^2 b \propto \frac{\sigma_1 \sqrt{c} l^2 b}{l^2} \propto \sigma_{fs} b t^2 \]

\[ (\sigma_f^*)_1 \propto \sigma_{fs} \left( \frac{l}{d} \right)^2 \frac{\sqrt{l}}{l_c} \]

\[ K_{IC}^* = \sigma_f^* \frac{\pi l^2}{l_c} = C \sigma_{fs} \left( \frac{l}{l_c} \right)^2 \sqrt{l} \]

depends on cell size, \( l \)!

\( C = \) constant

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Summary: hexagonal honeycombs, in-plane properties

- linear elastic moduli: \( E_1^* \), \( E_2^* \), \( D_{12}^* \), \( G_{12}^* \)
- plateau stresses \( (\sigma_{el}^*)_2 \) (compression)
  - \( \sigma_{pl}^* \) plastic yield
  - \( \sigma_{el}^* \) brittle crushing
- fracture toughness \( K_{IC}^* \) (tension)
  - \( K_{IC}^* \) brittle fracture
Honey combs: In-plane behaviour - triangular cells

- triangulated structures - trusses
- can analyse as pin-jointed (no moment @ joints)
- forces in members all axial (no bending)
- If joints fixed + include bending, difference ~ 2%
- force in each member proportional to P

\[
\sigma \propto \frac{P}{A} \quad \epsilon \propto \frac{\sigma}{E} \quad \delta \propto \frac{P l}{A E_s} \quad (\text{axial shortening: Hooker's law})
\]

\[
E^* \propto \frac{E}{E_s} \propto \frac{P}{b \ell} \propto \frac{b l E_s}{P l} \propto E_s \left( \frac{t}{l} \right)
\]

\[
E^* = C E_s \left( \frac{t}{l} \right)
\]

exact calculation: \( E^* = 1.15 E_s \left( \frac{t}{l} \right) \) for equilateral triangle.
Square and Triangular Honeycombs
