Open-cell foams

- stress-strain curves: deformation & failure mechanisms
- compression - 3 regimes - linear elastic - bending
  - stress plateau - cell collapse by buckling, yielding, crushing
- tension - no buckling
  - yielding can occur
  - brittle fracture

Linear elastic behaviour

- initial linear elasticity - bending of cell edges (small t/l)
- as t/l ↑, axial deformation becomes more significant
- consider dimensional argument, which models mechanism of deformation & failure, but not cell geometry
- consider cubic cell, square cross-section members of area $t^2$, length $l$
Foams: Bending, Buckling

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Foams: Plastic Hinges

Foams: Cell Wall Fracture

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regardless of specific cell geometry chosen:

\[ \rho^* / \rho_s \propto (t/l)^2 \]
\[ I \propto t^4 \]
\[ \sigma \propto F/l^2 \]
\[ E \propto \delta / l \]
\[ J \propto F l^3 / E_s I \]

\[ E^* \propto \frac{E}{E_s} \frac{F}{l^2} \frac{l}{\delta} \propto \frac{F}{E_s} \frac{E_s t^4}{l^3} \propto E_s (\frac{t}{l})^4 \propto E_s \left( \frac{\rho^*}{\rho_s} \right)^2 \]

\[ E^* = C_1 E_s \left( \frac{\rho^*}{\rho_s} \right)^2 \]

\[ C_1 \] includes all geometrical constants.

Data: \( C_1 \approx 1 \)

- data suggest \( C_1 = 1 \)
- analysis of open cell tetrahexahedral cells with Plateau borders gives \( C_1 = 0.98 \)
- shear modulus \( G^* = C_2 E_s \left( \frac{\rho^*}{\rho_s} \right)^2 \)

\[ C_2 \approx \frac{3}{8} \] if foam isotropic

\[ G = \frac{E}{2(1 + \nu)} \]

- Poisson's ratio \( \nu^* = \frac{E}{2G} - 1 = \frac{C_1}{2C_2} - 1 = \text{constant, independent of } E_s, t/e \)

\[ \nu^* = C_3 \]

(analogous to hexagonal cells in-plane)
Foam: Edge Bending

Poisson's ratio

- can make negative Poisson's ratio foams
- Invert cell angles (analogous to honeycomb)
- e.g. thermo-plastic foams - load hydrostatically
  + heat to $T > T_g$, then cool + release load
  so that edges of cell permanently point inward.

Closed-cell foams

- edge bending as for open-cell foams
- also: face stretching + gas compression
- polymer foams: surface tension draws material to edges during processing
  + define $t_e, t_f$ in figure
- apply $F$ to the cubic structure
Negative Poisson’s Ratio

- External work done $\propto F\delta$
- Internal work bending edges $\propto \frac{F\varepsilon_e e^2}{\varepsilon_e} \propto \frac{E_s I}{l^3} \delta^2$
- Internal work stretching faces $\propto \sigma_f E_f V_f \propto E_s E_f \varepsilon_f V_f \propto E_s (\varepsilon_f l) \tau_f l^2$

$$\therefore F\delta = \alpha E_s \varepsilon_t e^4 \delta^2 + \beta E_s (\varepsilon_f l)^2 \tau_f l^2$$

$$E^* \propto \frac{F}{l^2} \frac{l}{\delta} = \frac{E}{l} \Rightarrow F \propto E^* \delta l$$

$$\therefore E^* \delta^2 l = \alpha E_s \varepsilon_t e^4 \delta^2 + \beta E_s (\varepsilon_f l)^2 \tau_f l^2$$

$$E^* = \alpha E_s \left(\frac{e}{l}\right)^4 + \beta \frac{E_s (\varepsilon_f l)^2}{l^2}$$

Note: open cells, uniform $t$:
$$\frac{\rho^*}{\rho_0} \propto \left(\frac{t}{l}\right)^2$$

Closed cells, uniform $t$:
$$\frac{\rho^*}{\rho_0} \propto \left(\frac{t}{e}\right)$$

$$\frac{E^*}{E_s} = C_1 \phi^2 \left(\frac{\rho^*}{\rho_0}\right)^2 + C_1 (1-\phi) \frac{f^*}{\rho_0}$$
Closed-Cell Foam

Closed cell foams - gas within cell may also contribute to $E^*$

- Cubic element of foam of volume $V_o$
- Under uniaxial stress, axial strain in direction of stress is $\varepsilon$
- Deformed volume $V$ is:

$$
\frac{V}{V_o} = 1 - \varepsilon (1-2\nu^*)
$$

Taking compressive strain as positive, neglecting $\varepsilon^2, \varepsilon^3$ terms

$$
\frac{V_g}{V_o^*} = \frac{1 - \varepsilon (1-2\nu^*) - \rho^*/\rho_s}{1 - \rho^*/\rho_s}
$$

$V_g =$ volume gas, $V_o^*$ = "" initially

- Boyles law: $pV_g = p_o V_o^*$

$p =$ pressure after strain $\varepsilon$
$p_o = $ pressure initially

Pressure that must be overcome is $p' = p - p_o$

$$
p' = \frac{p_o \varepsilon (1-2\nu^*)}{1 - \varepsilon (1-2\nu^*) - \rho^*/\rho_s}
$$

- Contribution of gas compression to the modulus, $E^*$:

$$
E_g^* = \frac{dp'}{d\varepsilon} = \frac{p_o (1-2\nu^*)}{1 - \rho^*/\rho_s}
$$
\[ V_0 = l_0^3 \]
\[ \varepsilon_1 = \frac{l_1 - l_0}{l_0} \]
\[ l_1 = l_0 + \varepsilon_1 l_0 = l_0 (1 + \varepsilon_1) \]

\[ V = l_1 l_2 l_3 \]
\[ \varepsilon_2 = \frac{l_2 - l_0}{l_0} \]
\[ l_2 = l_0 + \varepsilon_2 l_0 = l_0 (1 + \varepsilon_2) \]
\[ \varepsilon_2 = -\varepsilon_1 \]

\[ G_3 = l_0 (1 - \varepsilon_1) \]

\[ V = l_1 l_2 l_3 = l_0 (1 + \varepsilon_1) l_0 (1 - \varepsilon_1) l_0 (1 - \varepsilon_2) = l_0^2 (1 + \varepsilon_1) (1 - \varepsilon_1) (1 - \varepsilon_2) \]

\[ \frac{V}{V_0} = \frac{l_0^3 (1 + \varepsilon_1) (1 - \varepsilon_1) (1 - \varepsilon_2)}{l_0^3} = (1 + \varepsilon_1) (1 - 2\varepsilon_1 + \varepsilon^2_1) \]

\[ = (1 - 2\varepsilon + \varepsilon^2) + \varepsilon = -2\varepsilon^2 + 3\varepsilon - 1 \]

\[ = 1 - \varepsilon (1 - 2\varepsilon) \]

\[ \text{taking comp. as +} \]

\[ = 1 - \varepsilon (1 - 2\varepsilon) \]
\[ p' = p - p_0 \]

\[ p = \frac{p_0 V_e}{V_g} \]

\[ p' = p - p_0 = p_0 \frac{V_e}{V_g} - p_0 = p_0 \left( \frac{V_e}{V_g} - 1 \right) + p_0 \left( \frac{V_g - V_e}{V_g} \right) \]

\[ = p_0 \left[ \frac{1 - \rho \psi_3}{1 - \epsilon (1 - 2 \psi^*) - \rho^* \psi_3} - 1 \right] \]

\[ = p_0 \left[ \frac{\rho - \rho \psi_3}{1 - \epsilon (1 - 2 \psi^*) - \rho^* \psi_3} \right] \]

\[ = p_0 \left[ \frac{\epsilon (1 - 2 \psi^*)}{1 - \epsilon (1 - 2 \psi^*) - \rho^* \psi_3} \right] \]
Closed cell foam

\[
\frac{E^*}{E_S} = \phi^2 \left( \frac{\rho^*}{\rho_S} \right)^2 + (1-\phi) \left( \frac{\rho^*}{\rho_S} \right) + \frac{P_0}{E_S} \frac{1-2\nu^*}{(1-\rho^*/\rho_S)}
\]

edge bending  face stretching  gas compression

- note: If \( P_0 = P_{atm} = 0.1 \) MPa, gas compression term negligible, except for closed-cell elastomeric foams
- gas comp. can be significant if \( P_0 \gg P_{atm} \); also modifies shape of stress plateau in elastomeric closed-cell foams

Shear modulus: edge bending, face stretching; shear \( \Delta V = 0 \) gas contrib. = 0

\[
\frac{G^*}{G_S} = \frac{3}{8} \left[ \phi^2 \left( \frac{\rho^*}{\rho_S} \right)^2 + (1-\phi) \left( \frac{\rho^*}{\rho_S} \right) \right] \quad \text{(isotropic foam)}
\]

Poisson's ratio = f (cell geometry only) \( \nu^* \approx \frac{1}{3} \)

Comparison with data
- data for polymers, glasses, elastomers
- \( E_S, \rho_S \) Table 5.1 in book
- open cells - open symbols
- closed cells - filled symbols
Young’s Modulus

Shear Modulus

Poisson’s Ratio

Non-linear elasticity

Open cells:

\[ P_{c} = \frac{n^2 \pi^2 E_s I}{l^2} \]

\[ \sigma_{el}^* \propto \frac{P_{c}}{l^2} \rho_s \frac{E_s}{(l^2)} \]

\[ \sigma_{el}^* = C_4 E_s \left( \frac{l^2}{\rho_s} \right)^2 \]

Data: \( C_4 \approx 0.05 \), corresponds to strain when buckling initiates, since \( E^* = E_s \left( \frac{l^2}{\rho_s} \right)^2 \)

Closed cells

- Often small compared to \( t_e \) (surface tension in processing) - contrib small
- If \( p_o \gg p_{atm} \), cell walls pretensioned; buckling stress has to overcome this

\[ \sigma_{el}^* = 0.05 E_s \left( \frac{l^2}{\rho_s} \right)^2 + p_o - p_{atm} \]

- Post-collapse behaviour - stress plateau rises due to gas compression (if faces don't rupture) \( \nu^* = 0 \) in post-collapse regime

\[ p' = \frac{p_o \varepsilon (1-2\nu^*)}{1 - \varepsilon (1-2\nu^*)} = \frac{p_o \varepsilon}{1 - \varepsilon - \rho^*/\rho_s} \]

\[ \sigma_{post-collapse}^* = 0.05 E_s \left( \frac{l^2}{\rho_s} \right)^2 + \frac{p_o \varepsilon}{1 - \varepsilon - \rho^*/\rho_s} \]
Elastic Collapse Stress

Elastic Collapse Stress

Post-collapse stress strain curve

Plastic collapse

Open cells

- failure when $M = M_p$
- $M_p \propto \sigma_{ys} t^3$  $M \propto \sigma^* p d^3$
- $\sigma^* p d = C_s \sigma_{ys} (p^*/\beta)^{3/2}$  $C_s \approx 0.3$, from data.
- elastic collapse precedes plastic collapse if $\sigma^* e_l < \sigma^* p d$

\[
0.05\ E_s\ (p^*/\beta)^2 \leq 0.3\ \sigma_{ys}\ (p^*/\beta)^{3/2}
\]

\[
(p/\beta) \leq 36\ (\sigma_{ys}/E_s)^2
\]

rigid polymers $p^*/\beta < 0.04\ (\sigma_{ys}/E_s)$

metals $p^*/\beta < 10^{-5}\ (\sigma_{ys}/E_s)$

Closed cells

- including all terms $\sigma^* p d = C_s \sigma_{ys} \phi (p^*/\beta)^{3/2} + C_s \sigma_{ys} (1-\phi)(p^*/\beta) + P_{edge\ bending}$
- but in practice, faces often rupture around $\sigma^* p d$, often $\sigma^* p d = 0.5(p^*/\beta)^{3/2} \sigma_{ys}$
Plastic Collapse Stress

Plastic Collapse Stress

Brittle crushing strength

Open cells

- failure when $M = M_f$
  
  $$\sigma_{cr}^* = C_6 \sigma_{fs} (\rho^*/\rho_0)^{3/2}$$

  
  $C_6 \approx 0.2$

- Densification strain, $E_p$
  
  - at large comp. strain, cell walls begin to touch, $\sigma$-$\varepsilon$ rises steeply
  
  - $E^* = E_s$; $\sigma$-$\varepsilon$ curve looks vertical, at limiting strain
  
  - Might expect $E_p = 1 - \rho^*/\rho_0$

  - Walls jam together at slightly smaller strain than this:
    
    $$E_p = 1 - 1.4 \rho^*/\rho_0$$
Densification Strain

\[ \varepsilon_D = 1 - 1.4 \frac{\rho^*_s}{\rho_s} \]

Fracture toughness

Open cells: crack length $2a$, local stress $\sigma_l$, remote stress $\sigma_\infty$

$$\sigma_l = C \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}}$$

- next unbroken cell wall a distance $r \approx l/2$ a load of crack tip subject to a force (integrating stress over next cell)

$$F \propto \sigma_l l^2 \propto \sigma_\infty \frac{\sqrt{a}}{l} \frac{l^2}{l^2}$$

- edges fail when applied moment, $M = \text{fracture moment, } M_f$

$$M_f \propto \sigma_{fs} t^3$$

$$M \propto Fl^* \propto \sigma_\infty \left(\frac{a}{l}\right)^{1/2} l^3$$

$$M = M_f = \sigma_\infty \left(\frac{a}{l}\right)^{1/2} l^3 \propto \sigma_{fs} t^3$$

$$\sigma_\infty \propto \sigma_{fs} \left(\frac{l}{a}\right)^{1/2} \left(\frac{t}{l}\right)^3$$

$$K_{IC}^* = \sigma_\infty \sqrt{\pi a} = C_8 \sigma_{fs} \sqrt{\pi l} \left(\frac{\rho}{\rho_s}\right)^{3/2}$$

Data: $C_8 \approx 0.65$
Fracture Toughness
Fracture Toughness
