Open-cell foams

- Stress-Strain curve: deformation and failure mechanisms
- Compression - 3 regimes - linear elastic - bending
  - stress plateau - cell collapse by buckling yielding crushing
  - densification
- Tension - no buckling
  - yielding can occur
  - brittle fracture

Linear elastic behavior

- Initial linear elasticity - bending of cell edges (small $t/l$)
- As $t/l$ goes up, axial deformation becomes more significant
- Consider dimensional argument, which models mechanism of deformation and failure, but not cell geometry
- Consider cubic cell, square cross-section members of area $t^2$, length $l$
Foams: Bending, Buckling

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Foams: Plastic Hinges

Foams: Cell Wall Fracture

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Regardless of specific geometry chosen:

\[ \frac{\rho^*}{\rho_s} \propto (t/l)^2 \quad I \propto t^4 \]

\[ \sigma \propto F/l^2 \quad \epsilon \propto \delta/l \quad \delta \propto \frac{Fl^3}{E_s I} \]

\[
E^* \propto \frac{\sigma}{\epsilon} \propto \frac{F}{l^2} \frac{l}{\delta} \propto \frac{F}{l} \frac{E_s t^4}{Fl^3} \propto E_s \left( \frac{t}{l} \right)^4 \propto E_s \left( \frac{\rho^*}{\rho_s} \right)^2
\]

\[ E^* = C_1 E_s \left( \frac{\rho^*}{\rho_s} \right)^2 \]

\( C_1 \) includes all geometrical constants

Data: \( C_1 \approx 1 \)

- Data suggests \( C_1 = 1 \)

- Analysis of open cell tetrakaidecahedral cells with Plateau borders gives \( C_1 = 0.98 \)

- Shear modulus \( G^* = C_2 E_s \left( \frac{\rho^*}{\rho_s} \right)^2 \)

\( C_2 \sim 3/8 \) if foam is isotropic

Isotropy: \( G = \frac{E}{2(1 + \nu)} \)

- Poisson’s ratio: \( \nu^* = \frac{E}{2G} - 1 = \frac{C_1}{2C_2} - 1 = \) constant, independent of \( E_s, \ t/l \)

\[ \nu^* = C_3 \] (analogous to honeycombs in-plane)
Foam: Edge Bending

Poisson’s ratio

- Can make negative Poisson’s ratio foams

- Invert cell angles (analogous to honeycomb)

- Eg. thermoplastic foams - load hydrostatically and heat to $T > T_g$, then cool and release load so that edges of cell permanently point inward

Closed-cell foams

- Edge bending as for open cell foams

- Also: face stretching and gas compression

- Polymer foams: surface tension draws material to edges during processing
  - define $t_e, t_f$ in figure

- Apply $F$ to the cubic structure
Negative Poisson’s Ratio

• External work done $\propto F\delta$.
• Internal work bending edges $\propto \frac{E_s}{\delta_e} \delta_e^2 \propto \frac{E_s l}{l^3} \delta^2$
• Internal work stretching faces $\propto \sigma_f \epsilon_f \nu_f \propto E_s \epsilon_f^2 \nu_f \propto E_s \left(\frac{\delta}{l}\right)^2 t_f l^2$

\[ F\delta = \alpha \frac{E_s t_e^4}{l^3} \delta^2 + \beta E_s \left(\frac{\delta}{l}\right)^2 t_f l^2 \]

\[ E^* \propto \frac{F}{l^2} \delta \rightarrow F \propto E^* \delta l \]

\[ E^* \delta^2 l = \alpha \frac{E_s t_e^4}{l^3} \delta^2 + \beta E_s \left(\frac{\delta}{l}\right)^2 t_f l^2 \]

\[ E^* = \alpha E_s \left(\frac{t_e}{l}\right)^4 + \beta E_s \left(\frac{t_f}{l}\right) \]

Note: Open cells, uniform $t$:
\[ \frac{\rho^*}{\rho_s} \propto (t/l)^2 \]

Closed cells, uniform $t$:
\[ \frac{\rho^*}{\rho_s} \propto (t/l) \]

If $\phi$ is volume fraction of solid in cell edges:
\[ \frac{t_e}{l} = C\phi^{1/2}(\rho^*/\rho_s)^{1/2} \]
\[ \frac{t_f}{l} = C'(1 - \phi) \left(\frac{\rho^*}{\rho_s}\right) \]

\[ \frac{E^*}{E_s} = C_1 \phi^2 \left(\frac{\rho^*}{\rho_s}\right)^2 + C_1' (1 - \phi) \rho^*/\rho_s \]
Closed-Cell Foam

Cell Membrane Stretching

Closed cell foams - gas within cell may also contribute to $E^*$

- Cubic element of foam of volume $V_0$
- Under uniaxial stress, axial strain in direction of stress is $\epsilon$
- Deformed volume $V$ is:

$$
\frac{V}{V_0} = 1 - \epsilon(1 - 2\nu^*) \\
\frac{V_g}{V_g^0} = \frac{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s}{1 - \rho^*/\rho_s}
$$

taking compressive strain as positive, neglecting $\epsilon^2, \epsilon^3$ terms

$V_g = \text{volume gas}$

$V_g^0 = \text{volume gas initially}$

- Boyle’s law: $pV_g = p_0V_g^0$

$p = \text{pressure after strain } \epsilon$

$p_0 = \text{pressure initially}$

- Pressure that must be overcome is $p' = p - p_0$:

$$
p' = \frac{p_0 \epsilon (1 - 2\nu^*)}{1 - \epsilon (1 - 2\nu^*) - \rho^*/\rho_s}
$$

- Contribution of gas compression to the modulus $E^*$:

$$
E_g^* = \frac{dp'}{d\epsilon} = \frac{p_0(1 - 2\nu^*)}{1 - \rho^*/\rho_s}
$$
\[ V_0 = l_0^3 \quad \epsilon_1 = \frac{l_1 - l_0}{l_0} \quad \rightarrow \quad l_1 = l_0 + \epsilon_1 l_0 = l_0(1 + \epsilon_1) \]

\[ V = l_1 l_2 l_3 \quad \epsilon_2 = \frac{l_2 - l_0}{l_0} \quad \rightarrow \quad l_2 = l_0 + \epsilon_2 l_0 \quad \quad \nu = -\frac{\epsilon_2}{\epsilon_1} \]
\[ = l_0 - \nu \epsilon_1 l_0 \quad \epsilon_2 = -\nu \epsilon_1 \]
\[ = l_0(1 - \nu \epsilon_1) \]
\[ \epsilon_3 = l_0(1 - \nu \epsilon_1) \]

\[ V = l_1 l_2 l_3 = l_0(1 + \epsilon_1) l_0(1 - \nu \epsilon_1) l_0(1 - \nu \epsilon_1) = l_0^3(1 + \epsilon_1)(1 - \nu \epsilon_1)^2 \]
\[ \frac{V}{V_0} = \frac{l_0^3(1 + \epsilon)(1 - \nu \epsilon)^2}{l_0^3} = (1 + \epsilon)(1 - 2\nu \epsilon + \nu^2 \epsilon^2) \]
\[ = (1 - 2\nu \epsilon + \nu^2 \epsilon^2) + \epsilon - 2\nu \epsilon^2 + \nu^2 \epsilon^3 \]
\[ = 1 - \epsilon + 2\nu \epsilon \]
\[ = 1 - \epsilon(1 - 2\nu) \]
\[ p' = p - p_0 \]

\[ p = \frac{p_0 V'_g}{V_g} \]

\[ p' = p - p_0 = \frac{p_0 V'_g}{V_g} - p_0 = p_0 \left( \frac{V^0_g}{V_g} - 1 \right) \]

\[ = p_0 \left[ \frac{1 - \rho^*/\rho_s}{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s} - 1 \right] \]

\[ = p_0 \left[ \frac{1 - \rho^*/\rho_s (1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s)}{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s} \right] \]

\[ = p_0 \left[ \frac{\epsilon(1 - 2\nu^*)}{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s} \right] \]
Closed cell foam

\[
\frac{E^*}{E_s} = \phi^2 \left( \frac{\rho^*}{\rho_s} \right)^2 + (1 - \phi) \left( \frac{\rho^*}{\rho_s} \right) + \frac{p_0 (1 - 2\nu^*)}{E_s (1 - \rho^*/\rho_s)}
\]

edge bending  face stretching  gas compression

- Note: if \( p_0 = p_{\text{atm}} = 0.1 \) MPa, gas compression term is negligible, except for closed-cell elastomeric foams

- Gas compression can be significant if \( p_0 >> p_{\text{atm}} \); also modifies shape of stress plateau in elastomeric closed-cell foams

Shear modulus: edge bending, face stretching; shear \( \Delta V = 0 \) gas contribution is 0

\[
\frac{\epsilon^*}{E_s} = \frac{3}{8} \left[ \phi^2 \left( \frac{\rho^*}{\rho_s} \right)^2 + (1 - \phi) \left( \frac{\rho^*}{\rho_s} \right) \right] \quad \text{(isotropic foam)}
\]

Poison’s ratio = \( f \) (cell geometry only) \( \nu^* \approx 1/3 \)

**Comparison with data**

- Data for polymers, glasses, elastomers
- \( E_s, \rho_s \) - Table 5.1 in the book
- Open cells — open symbols
- Closed cells — filled symbols
Young’s Modulus

Shear Modulus

Poisson’s Ratio

Non-linear elasticity

Open cells:

\[ P_{cr} = \frac{n^2 \pi^2 E_s I}{l^2} \]

\[ \sigma_{el}^* \propto \frac{P_{cr}}{l^2} \propto E_s \left( \frac{t}{l} \right)^4 \]

\[ \sigma_{el}^* = C_4 E_s \left( \frac{\rho^*}{\rho_s} \right)^2 \]

Data: \( C_4 \approx 0.05 \), corresponds to strain when buckling initiates, since \( E^* = E_s \left( \frac{\rho^*}{\rho_s} \right)^2 \)

Closed cells:

- \( t_f \) often small compared to \( t_e \) (surface tension in processing) - contribution small
- If \( p_0 >> p_{atm} \), cell walls pre-tensioned, bucking stress has to overcome this

\[ \sigma_{el}^* = 0.05 E_s \left( \frac{\rho^*}{\rho_s} \right)^2 + p_0 - p_{atm} \]

- Post-collapse behavior - stress plateau rises due to gas compression (if faces don’t rupture) \( \nu^* = 0 \) in post-collapse regime

\[ p' = \frac{p_0 \epsilon (1 - 2\nu^*)}{1 - \epsilon (1 - 2\nu^*) - \rho^*/\rho_s} = \frac{p_0 \epsilon}{1 - \epsilon - \rho^*/\rho_s} \]

\[ \sigma_{post-collapse}^* = 0.05 E_s \left( \frac{\rho^*}{\rho_s} \right)^2 + \frac{p_0 \epsilon}{1 - \epsilon - \rho^*/\rho_s} \]
Elastic Collapse Stress

Elastic Collapse Stress

Post-collapse stress strain curve

Plastic collapse

Open cells:

• Failure when $M = M_p$

• $M_p \propto \sigma_{ys} t^3 \quad M \propto \sigma_{pl}^3$

\[ \sigma_{pl}^* = C_5 \sigma_{ys} (\rho^*/\rho_s)^{3/2} \quad C_5 \sim 0.3 \text{ from data.} \]

• Elastic collapse precedes plastic collapse if $\sigma_{el}^* \leq \sigma_{pl}^*$

\[ 0.05 E_s (\rho^*/\rho_s)^2 \leq 0.3 \sigma_{ys} (\rho^*/\rho_s)^{3/2} \quad \text{rigid polymers} \quad (\rho^*/\rho_s)_{cr} < 0.04 \left( \frac{\sigma_{ys}}{E_s} \right) \sim \frac{1}{30} \]

\[ (\rho^*/\rho_s)_{critical} \leq 36 (\sigma_{ys}/E_s)^2 \quad \text{metals} \quad (\rho^*/\rho_s)_{cr} < 10^{-5} \left( \frac{\sigma_{ys}}{E_s} \right) \sim \frac{1}{1000} \]

Closed cells:

• Including all terms: $\sigma_{pl}^* = C_5 \sigma_{ys} \left( \frac{\rho^*/\rho_s}{\phi} \right)^{3/2} + C_5' \sigma_{ys} (1 - \phi) \left( \frac{\rho^*/\rho_s}{\phi} \right) + p_0 - p_{atm}$

  \[ \uparrow \text{edge bending} \quad \uparrow \text{face stretching} \]

• But in practice, faces often rupture around $\sigma_{pl}^*$ - often $\sigma_{pl}^* = 0.3 (\rho^*/\rho_s)^{3/2} \sigma_{ys}$
Plastic Collapse Stress

Plastic Collapse Stress

Brittle crushing strength

Open cells:

- Failure when $M = M_f$, $M \propto \sigma_{cr}^* l^2$, $M_f \propto \sigma_{fs} t^3$

\[
\sigma_{cr} = C_6 \sigma_{fs} \left( \frac{\rho^*}{\rho_s} \right)^{3/2} \quad C_6 \approx 0.2
\]

Densification strain, $\epsilon_D$:

- At large comp. strain, cell walls begin to touch, $\sigma - \epsilon$ rises steeply
- $E^* \to E_s$; $\sigma - \epsilon$ curve looks vertical, at limiting strain
- Might expect $\epsilon_D = 1 - \rho^*/\rho_s$
- Walls jam together at slightly smaller strain than this:

\[
\epsilon_D = 1 - 1.4 \frac{\rho^*}{\rho_s}
\]
Densification Strain

\[ \epsilon_D = 1 - 1.4 \frac{\rho^*}{\rho_s} \]

Fracture toughness

Open cells: crack length $2a$, local stress $\sigma_l$, remote stress $\sigma^\infty$

$$\sigma_l = \frac{C \sigma^\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \quad \text{a distance } r \text{ from crack tip}$$

- Next unbroken cell wall a distance $r \approx l/2$, a head of crack tip subject to a force (integrating stress over next cell)

$$F \propto \sigma_l l^2 \propto \sigma^\infty \sqrt{\frac{\pi}{l}} l^2$$

- Edges fail when applied moment, $M = \text{fracture moment, } M_f$

$$M_f = \sigma_{fs} t^3$$

$$M \propto F l \propto \sigma^\infty \left(\frac{a}{l}\right)^{1/2} l^3 \quad M = M_f \rightarrow \sigma^\infty \left(\frac{a}{l}\right)^{1/2} l^3 \propto \sigma_{fs} l^3$$

$$\sigma^\infty \propto \sigma_{fs} \left(\frac{1}{a}\right)^{1/2} \left(\frac{t}{l}\right)^3$$

$$K_{IC}^* = \sigma^\infty \sqrt{\pi a} = C_8 \sigma_{fs} \sqrt{\pi l (\rho^*/\rho_s)}^{3/2}$$

Data: $C_8 \sim 0.65$
Fracture Toughness

Fracture Toughness
