The Stress Tensor

Consider a force-density vector \( \bar{F} \) (force per unit area) acting on the external surface \( ABC \) of a volume element \( OABC \). We assume this volume element is in equilibrium (no linear or angular acceleration). The external force density \( \bar{F} \) that pushes on the solid must, therefore, be balanced by the forces inside the volume element pushing back against the force.

Let us set up relations that balance external (and internal) forces in the \( x_1, x_2, x_3 \) directions as \( \bar{F} = F_1 \hat{x}_1, F_2 \hat{x}_2, F_3 \hat{x}_3 \) is a force density, we must multiply by an area to get force.

Along \( x_1 \),

\[ \bar{F}_1 = \bar{T}_{11} \text{AREA BOC} + \bar{T}_{12} \text{AREA AOC} + \bar{T}_{13} \text{AREA AOB} \]

We have put in quantities \( \bar{T}_{ij} \), forces per unit area, with two subscripts, the first is the direction in which the force density acts, the second specifies the normal to the internal surface on which the force density acts, thus \( \bar{T}_{ij} \) is the force per unit area acting along \( x_i \) on an internal surface whose normal is \( \hat{x}_j \).

Let us divide through by \( \text{AREA ABC} \)

\[ \bar{F}_1 = \bar{T}_{11} \frac{\text{AREA BOC}}{\text{AREA ABC}} + \bar{T}_{12} \frac{\text{AREA AOC}}{\text{AREA ABC}} + \bar{T}_{13} \frac{\text{AREA AOB}}{\text{AREA ABC}} \]

If we have an area \( A \) in one plane and project it onto a surface in a plane that makes an angle \( \phi \) with respect to the first, the new area \( A' = A \cos \phi \)

Note that \( \phi \) is the same thing as the angle between the normal to the original surface and the normal to the new surface \( \bar{T}_1 \). Looking at our equation for \( F_1 \) above

\[ \frac{\text{AREA BOC}}{\text{AREA ABC}} = \cos \phi \text{ angle between the normal to the surface ABC and the normal to BOC, which is } \hat{x}_1 \text{.} \]

This specifies the direction cosine of the normal to the surface \( ABC \).

Similarly

\[ \frac{\text{AREA AOC}}{\text{AREA ABC}} = \cos \theta \text{ a direction cosine of the normal to } \text{SURFACE ABC} \]

\[ \cos \phi \text{ and } \cos \theta \]

It follows that the remaining equations, balancing forces in the \( x_2 \) and \( x_3 \) directions are

\[ \begin{align*}
F_2 &= T_{21} l_1 + T_{22} l_2 + T_{23} l_3 \\
F_3 &= T_{31} l_1 + T_{32} l_2 + T_{33} l_3
\end{align*} \]

The nine coefficients \( T_{ij} \) are called the elements of stress.

\( \bar{F} = F_1 \hat{x}_1, F_2 \hat{x}_2, F_3 \hat{x}_3 \) is a vector, we know the law of transformation of a vector upon change of coordinate system.

\( c_{ij} \) are the elementary coefficients.

The elements \( \bar{\lambda}_i \) constitute a vector — this is \( \bar{T}_1 \), a unit vector normal to the external surface \( ABC \). The components \( \bar{\lambda}_i' = \bar{\lambda}_i \) thus become \( \bar{\lambda}_i' = c_{ij} \bar{\lambda}_j \)

The coefficients \( T_{ij} \) must transform like a tensor, and accordingly \( T_{ij} \) is a tensor.
The sign of an element of stress can be + or −, we define the following convention. A positive element of stress is when the material outside the volume element acts on the material inside the volume element to produce a force per unit area in the +x₃ direction on a surface whose normal is +x₁ (or -x₄ and -x₅).

If the solid is in equilibrium, the stress tensor must be symmetric.

Consider the elements of stress $\sigma_{12}$ and $\sigma_{21}$.

$\sigma_{12}$ and the same $\sigma_{21}$ on the opposite side of the volume element create a couple that would act to produce an angular acceleration. $\sigma_{21}$ is the only couple that can balance $\sigma_{12}$.

$\therefore \sigma_{12} = \sigma_{21}$ and $\sigma_{ij} = \sigma_{ji}$ for equilibrium.

As stress is a second-rank tensor, we can define a stress quadric:

\[ \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \]

Everything that we have said about the representation quadric applies to the stress quadric.

(a) Reference axes can be changed: $x, y, z \rightarrow x', y', z'$ to have the tensor in a diagonal form.

(b) Using $\sigma_{ij} \lambda_i \lambda_j = 0$, we can compute the value of stress in a particular direction. We'll have to think a moment to see what that means! The applied vector $\mathbf{F}$ in $\mathbf{F} = \sigma_{ij} \mathbf{n}_j$ is the set of direction cosines of the normal to the surface. The "generalized displacement" is $\mathbf{u}$, the imposed force density.

What $\sigma_{ij} \lambda_i \lambda_j$ provides is $\frac{F_i}{\text{unit normal}} = \frac{F_i}{|F|}$. This is the force per unit area that is parallel to the normal to the surface that is transmitted across the surface in other words the tensile component of the transmitted stress.

(c) Some special forms of the stress tensor when in diagonal form:

- Uniaxial stress
- Biaxial stress
- Triaxial stress
- Hydrostatic pressure
- shear stress

Note:

Stress is a field tensor, not a property tensor!
**Strain**

Let us introduce the concept of strain with a one-dimensional example. Let us consider an elastic band attached to a wall. If we exert a force on the elastic band, point P at location x before the deformation undergoes a displacement $u$ to point $P'$ at $(x + u)$.

A point Q at distance $\Delta x$ from P is also displaced; it moves a distance $u + \alpha u$ and is thus displaced to $Q'(x + \Delta x + u + \alpha u)$.

If we plot $u$ as a function of distance along the elastic band, we find that $u$ will increase linearly with $x$. This is termed homogeneous deformation; the slope of the plot is linear, and $\frac{du}{dx} = e$.

Let us return to our points P and Q. It is not displacement that interests us as much as relative displacement as this causes changes in length and forces within the material.

\[
\begin{align*}
\alpha \text{ P}' & = Q' = \Delta x (1+e) \\
\text{Relative change of length is } \frac{\Delta x (1+e) - \Delta x}{\Delta x} = e = \frac{du}{dx} = \text{Strain (dimensionless)}
\end{align*}
\]

Before continuing, we should pause to note that the real world displays types of strain in adding to homogeneous strain.

**Extension to Three Dimensions**

In the above expression for one dimension, we started by saying $u = ex (\frac{dx}{dx} = e)$ but then showed that changes in length behave the same way $\frac{du}{dx} = e$ = fractional change in length of line segment.

What we essentially said was that $\bar{u}$ is proportional to $\bar{x}$.

To express the fractional change of length in different directions in three dimensions, let's assume that $\bar{u}$ is proportional to (but not necessarily parallel to) position in the material, $\bar{x}$.

We can then write:

\[
\begin{align*}
\bar{u}_1 &= \frac{du_1}{dx_1} x_1 + \frac{du_2}{dx_1} x_2 + \frac{du_3}{dx_1} x_3 \\
\bar{u}_2 &= \frac{du_1}{dx_2} x_1 + \frac{du_2}{dx_2} x_2 + \frac{du_3}{dx_2} x_3 \\
\bar{u}_3 &= \frac{du_1}{dx_3} x_1 + \frac{du_2}{dx_3} x_2 + \frac{du_3}{dx_3} x_3 \\
\text{or } \bar{u}_x &= \frac{du_i}{dx_j} x_j
\end{align*}
\]

Or in terms of differences in displacement:

\[
\begin{align*}
\Delta \bar{u}_1 &= \frac{\Delta u_1}{\Delta x_1} \Delta x_1 + \frac{\Delta u_1}{\Delta x_2} \Delta x_2 + \frac{\Delta u_1}{\Delta x_3} \Delta x_3 \\
\text{or } \Delta \bar{u}_x &= \frac{\Delta u_i}{\Delta x_j} \Delta x_j \\
\text{Let's define } \frac{\Delta u_i}{\Delta x_j} = e_{ij}
\end{align*}
\]
Let us consider a line segment $PQ$ of length $\Delta x_1$ that is initially parallel to $x_1$.

$$\begin{align*}
\Delta u_1 &= \frac{\partial u_1}{\partial x_1} \Delta x_1 = e_{11} \Delta x_1, \\
\Delta u_2 &= \frac{\partial u_2}{\partial x_1} \Delta x_1 = e_{21} \Delta x_1
\end{align*}$$

As initially parallel to $x_1$, means there is no contribution $\Delta x_2$, this keeps things simple.

The line segment $PQ = \Delta x_1$ has changed length and has rotated direction.

The change in length is $\sqrt{(\Delta x + \Delta u_1)^2 + \Delta u_2^2} \approx \Delta x + \Delta u_1$, or more rigorously:

$$\Delta x' = \sqrt{(\Delta x + \Delta u_1)^2 + \Delta u_2^2} \approx \Delta x + \Delta u_1$$

The line segment $PQ'$ has also rotated an amount $\phi$.

$$\tan \phi = \frac{\Delta u_2}{\Delta x + \Delta u_1} \approx \frac{\Delta u_2}{\Delta x} = e_{21} \approx \phi$$

Thus $e_{21}$ is the angle of rotation of line segment initially along $x_1$ in the direction of $x_2$.

If $e_{21}$ is small and $\phi \approx \Delta u_1/\Delta x_1$, a generalization of the angle $\phi$ is $e_{ij} = \frac{\partial u_i}{\partial x_j}$, the angle of rotation of line segment along $x_i$ in the direction of axis $x_j$.

**Proper definition of a satisfactory measure of strain**

Consider in a 2-dimensional space a "state of strain" in which $e_{12} = -e_{21} = 0$.

That would produce displacements in which a line $PQ_1$ initially along $x_1$ is rotated toward $x_2$ by an angle $\phi \approx e_{12}$.

Consider a line $PQ_2$ initially along $x_2$. If $e_{12}$ is $-e_{21}$, it rotates $\phi$ to a new location $PQ_2'$.

A line along $x_1$ toward $x_3$ then a negative $e_{31}$ rotates a line along $x_3$ away from $x_1$.

Would one call this a "deformation"? No! It looks pretty much like rotation about $x_3$ with out any strain at all.

Indeed, if you picked up what had been initially a cube of material, and wanted to place it in a position so that angles such as $e_{21}$ and $e_{12}$ really did provide a measure of deformation, you would probably, by instinct, place the deformed block relative to the reference axes such that $e_{ij} = e_{ij}'$. In other words, if $e_{21} + e_{12}$, you would rotate the body until it was...

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

$$\frac{1}{2}(e_{12} + e_{21})$$

$$\frac{1}{2}(e_{21} - e_{12})$$

$$\frac{1}{2}(e_{12} + e_{21}) + \frac{1}{2}(e_{21} - e_{12})$$
This is a new concept: Addition of Tensors!

Let us take the tensor $E_{ij}$ and divide it into two parts

$$E_{ij} = E_{ij} + W_{ij}$$

with

$$E_{ij} = \frac{1}{2} (e_{ij} + e_{ji})$$
$$W_{ij} = \frac{1}{2} (e_{ij} - e_{ji})$$

$$E_{ij} = E_{ij} + W_{ij} = \frac{1}{2} (2e_{ij})$$

The tensor $W_{ij}$ will, from the way in which it was defined, have $W_{ii} = 0$ and $W_{ij} = -W_{ji}$. It is an "antisymmetric tensor".

**Proof that pure rotation with no strain corresponds an anti-symmetric $E_{ij}$**

If $e_{ij}$ relates a displacement vector $U_{ij}$ to a position in the solid $R = (x_1, x_2, x_3)$ and $E_{ij}$ corresponds to pure rigid body rotation, then a characteristic of $U_{ij}$ is that it is everywhere $\perp R$, regardless of the coordinates $x_1, x_2, x_3$ to which $R$ extends.

For this to be true, $U_{ij} \cdot R = 0$ (note: strain is assumed to be small).

Then $U_{ij} = E_{ij} x_i$ and $R_i = x_i$.

Expanding the above expression:

$$e_{11} x_1^2 + e_{22} x_2^2 + e_{33} x_3^2 + (e_{12} + e_{21}) x_1 x_2 + (e_{13} + e_{31}) x_1 x_3 + (e_{23} + e_{32}) x_2 x_3 + (e_{12} + e_{23}) x_1 x_3 = 0$$

For this to be true for all $x_i$, the coefficients of $x_i x_j$ must individually vanish:

$$e_{11} = e_{22} = e_{33} = 0$$
$$e_{12} + e_{21} = e_{13} + e_{31} = e_{23} + e_{32} = 0$$

Therefore rigid body rotation corresponds to

$$\begin{bmatrix} 0 & e_{12} & e_{13} \\ -e_{12} & 0 & e_{23} \\ -e_{13} & -e_{23} & 0 \end{bmatrix}$$

QED.

To define true strain $E_{ij}$ - that is, real deformation, we therefore split a general $E_{ij}$ into two parts $E_{ij} = E_{ij} + W_{ij}$ with $E_{ij} = \frac{1}{2} (e_{ij} + e_{ji})$ and $W_{ij} = \frac{1}{2} (e_{ij} - e_{ji})$.

The final result is, for true deformation:

$$U_{ij} = E_{ij} x_i$$

$U_{ij}$ is a displacement vector, $x_i$ is a position vector.

$U_{ij}$ and $x_i$ both transform according to the law for 2nd rank tensors. Therefore $E_{ij}$ is a tensor, the strain tensor, of 2nd rank.

All of the properties that we have ascribed to tensors in previous discussion therefore apply to the strain tensor.

- The elements of strain depend on the coordinate system to which they are referred. Upon change of reference axes $x_i, x_j, x_k \rightarrow x'_i, x'_j, x'_k$ described by $[C_{ij}]$, $E'_{ij} = C_{ij}E_{ij}C_m$. 

The "value of strain" in a given direction defined by direction cosines $l_i$ is $E = E_{ij} l_i l_j$. What does it mean? Turning to our definition of value of a property in a given direction,

$$q_j = a_j; \quad a = a_j l_j l_j = \frac{q_{11}}{101}$$

In the present case $E = E_{11}$ — it is the component of $u$ in the direction of $\bar{u}$, or the tensile displacement per unit length of $\bar{u}$.

A strain quadratic $E_{ij} x_i x_j = 1$ can be defined and has all of the properties that we assigned to the representation quadratic — including those that are valid only for symmetric tensors (as we have defined $E_{ij}$ as symmetric).

The strain quadratic, accordingly, has a radius that is $1/\sqrt{6}$ in the direction of the radius.

The "radial normal" property works because $E_{ij}$ is symmetric.

$$\Delta u = E_{ij} \Delta x_j$$

The initial line segment is rotated and changes length $E_{ij} l_i l_j$ gives $\Delta u$ per unit $\Delta x$ and is called the stretch in that direction.

A strain tensor in general form can be converted to a diagonal form by finding the eigenvalues and eigenvectors and referring the tensor to the new principal axes.

Volume change associated with deformation

Consider a volume element, a box with edges $L_1, L_2, L_3$.

To keep things simple, let's assume that we have taken the reference axes along the principal axes at the strain quadratic so that the strain tensor is diagonal.

After deformation

$$
\begin{bmatrix}
L_1 & L_1 (1 + E_{11}) & L_1 (1 + E_{12}) \\
L_2 & L_2 (1 + E_{22}) & L_2 (1 + E_{23}) \\
L_3 & L_3 (1 + E_{33}) & L_3 (1 + E_{33})
\end{bmatrix}
\begin{bmatrix}
E_{11} & 0 & 0 \\
0 & E_{22} & 0 \\
0 & 0 & E_{33}
\end{bmatrix}
$$

Then, if $V = L_1 L_2 L_3$

$$V' = V + AV = L_1 (1 + E_{11}) L_2 (1 + E_{22}) L_3 (1 + E_{33})$$

$$= L_1 L_2 L_3 (1 + E_{11} + E_{22} + E_{33} + E_{12} + E_{13} + E_{23} + E_{33} + E_{22} + E_{33})$$

As strains are small, neglect higher-order terms.

$$V + AV \approx L_1 L_2 L_3 (1 + E_{11} + E_{22} + E_{33})$$

$$\Delta V = V + V(E_{11} + E_{22} + E_{33})$$

$$\frac{\Delta V}{V} = E_{11} + E_{22} + E_{33}$$

This is the fractional change in volume caused by deformation — called the dilation.
WE SHOWN THAT THE TRACTIONAL VOLUME CHANGE UPON DEFORMATION IS 

\[ \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \]  

BECAUSE \( \Delta V / V \) IS A SCALAR QUANTITY, WE WOULD NOT EXPECT 
THE DILATION TO CHANGE UPON CHANGE OF REFERENCE AXES. INDEED, IT DOES NOT. 
THIS SUM OF DIAGONAL ELEMENTS FOR A SECOND-RANK TENSOR IS A GENERALLY-DEFINED 
QUANTITY FOR TENSORS AND IS CALLED THE TRACE OF THE TENSOR. 

\[ \text{TRACE} = T = \alpha_{11} + \alpha_{22} + \alpha_{33} \]  

IT HAS THE PROPERTY THAT IT REMAINS INvariant 
UPON ANY CHANGE OF REFERENCE AXES. 

(THIS IS STRAIGHTFORWARD TO PROVE. ADD TOGETHER \( \alpha_{11} = C_{11} \alpha_1 \cos^2 \theta + 
C_{22} \alpha_2 \cos^2 \theta + C_{33} \alpha_3 \cos^2 \theta \) = \( (C_{11} \alpha_1 \cos^2 \theta + C_{22} \alpha_2 \cos^2 \theta + C_{33} \alpha_3 \cos^2 \theta) \) \alpha_2 \cos^2 \theta 

\[ = 0 \quad \exists \neq 0 \]  

FROM THE ORTHOGONALITY 
PROPERTIES OF THE 
DIRECTOR COSINE MATRIX 

\[ : T = \alpha_{11} + \alpha_{22} + \alpha_{33} \quad (\text{FSD}) \]  

SOME SPECIAL FORMS OF THE DIAGONALIZED STRAIN TENSOR.

PLANE STRAIN  

"PURE SHEAR" (SPECIAL CASE OF PLANE STRAIN)  

\[
\begin{bmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-G_{11} & 0 & 0 \\
0 & -G_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

ROTATE AXES 
BY 45° 

\[
\begin{bmatrix}
0 & e' & 0 \\
e' & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

NOTE THAT TRACE = 0. SHEAR CAUSES NO VOLUME CHANGE!

SOME OTHER NON-TENSOR DEFINITIONS INVOLVING STRAIN

"SIMPLE SHEAR" 

\[ \begin{align*}
\gamma &= \frac{1}{2} \alpha + \frac{1}{2} \beta \\
\alpha &= \frac{1}{2} \gamma \\
\beta &= \frac{1}{2} \gamma
\end{align*} \]

ONE SOMETIMES SEE REFERENCE TO THE "STRAIN ELLIPSOID" 

CONSIDER A SPECIMEN INITIALLY IN THE FORM OF A SPHERE OF UNIT RADIUS 

\[ x_1^2 + x_2^2 + x_3^2 = 1 \]

AFTER DEFORMATION (TAKING \( x'_1, x'_2, x'_3 \) ALONG PRINCIPAL AXES)

\[
\begin{align*}
x'_1 &= x_1 (1 + \varepsilon_{11}) \\
x'_2 &= x_2 (1 + \varepsilon_{22}) \\
x'_3 &= x_3 (1 + \varepsilon_{33})
\end{align*}
\]

THE EQUATION OF THE SURFACE AFTER DEFORMATION WILL BE 

\[
\begin{align*}
x'_1^2 &= \left( x_1 (1 + \varepsilon_{11}) \right)^2 \\
x'_2^2 &= \left( x_2 (1 + \varepsilon_{22}) \right)^2 \\
x'_3^2 &= \left( x_3 (1 + \varepsilon_{33}) \right)^2
\end{align*}
\]

\[ x'_1^2 + x'_2^2 + x'_3^2 = 1 \]

THIS IS THE "STRAIN ELLIPSOID" (IT IS NOT THE STRAIN QUADRIC ...?)