PROFESSOR: Any questions before I obliterate all this lovely geometry? No. OK. We handled them during intermission I guess. Let me do a couple more plane groups just very quickly to show you how they come out without going through all of the steps, because I think we've seen now what one has to do to derive these.

There are two that are left. One is a threefold axis, and if that's all the symmetry we've put into the lattice, we're combining a threefold rotation axis with a primitive lattice, and we know that this has to be hexagonal net. Because from our depth of experience, we know that is the shape of a lattice that is demanded by a threefold axis.

It has two translations, T1, that are identical in length. And this angle between them is exactly 120 degrees. That's what we saw a threefold access require.

So now what we are doing is putting in a threefold axis at one lattice point, and this means we are adding the operations A 2 pi over 3. A minus 2 pi over 3.

I'll choose to define a 120 degree rotation that way, and the operation A 2 pi, which is the identity operations. So I'm adding three operations.

Putting the threefold axis in at all these locations. They will all be translationally equivalent. The pattern of this particular plane group is going to be the pattern of a threefold axis. And so there will be one motif here, 120 degrees away. There will be a-- sorry to wipe out those operations-- 120 degrees.

There'll be another object located here. And 120 degrees again, there will be another object located here. So these guys will fall at the corner of an equilateral triangle.

And the same will be at the other corners of the net, and that, as I've said before, is the pattern of that plane group. And that's a plane group that we would call P3. Primitive lattice that has to be hexagonal, and what we've added to it is a threefold axis.

OK, let me down here just indicate the combinations that we would do. And I could do it up on top, but I can work with this translation as well, which is easier to reach at this late hour in the afternoon. Let me combine A 2 pi over 3 with this translation here. And that says we have to get an operation B 2 pi over 3. That is located at the original translation T times the cotangent of 1/2 of 2 pi over 3. And that's at 1/2 of the cotangent of 60 degrees.

And this turns out not to be any nice neat number like 0 or 1/2, but if you evaluate what the distance up along the perpendicular bisector is by this amount, where you come out is right in the center of this triangle. Trust me. A little bit of trigonometry will let you convince yourself of that.

So if I rotate 120 degrees. Bring this one to this one, and then translate over to here, the way the first one and the second one are related is by a 120 degree rotation about the center of this triangle. If I combine the operation A minus 2 pi over 3, with this translation, I go down a distance x of minus this amount and that puts me in the middle of the triangle that is directly below. And I can move that back up. And now I have all the operations of a threefold axis about this location.

I can do the same thing with the threefold axis at another lattice point. Another thing I could do is to just rotate this by 120 degrees. And by either route, you find there must be a threefold axis here. Notice that these threefold axes will once again, as advertised, simply take things that are at the different corners of the cell.

For example, this threefold axis will tell you how this motif is related to this one is related to this one. And so it goes through the other threefold axes as well. This threefold axis will tell you this one is related. No, I don't want to draw it in.

So this is P3, a pair of threefold axes in the centers of the triangle, and another one at the corner of the cell. And now I am going to do the remaining combination of a rotation axis with a lattice, so quickly, and it's going to take your breath away. And that is seemingly the most difficult and complex one of all.

This would be P6. Sixfold axis plus a primitive lattice. And we know it also has to be hexagonal. And that will be called P6. What we're adding to the lattice is a sixfold axis. And now what I'm going to do as a shortcut is to say a sixfold axis also contains all the operations of a threefold axis. So I can take P3 and drop it right on top of P6, and that's going to give me threefold axes here.

Sixfold axis not only contains 2 pi over 6 and 2 pi over 3, it also contains the operation 2 pi over 2. This is the operation of a twofold axis, and that says I have to, in addition to the two full rotation that sits here get twofold rotations in the middle of every one of the translations T2.

So actually, this is going to be P2 superimposed on P3. And that's going to give me these twofold axis and this threefold axis. And the only thing I have to do is to show you what A 2 pi over 6 combined with one of these translations is going to do. What is A 2 pi over 6 combined with this translation?

And it's going to be new operation B 2 pi over 6, and it's going to be located at 1/2 of T times the cotangent of 1/2 of 60 degrees, cotangent of 30 degrees. And the cotangent of 30 degrees is 2.

So this is going to be at 1/2-- no, what do I want to say? This is 30 degrees. This is one. This is two. Cotangent of 30 degrees is this over this, and that is-- no that's one.

AUDIENCE: It's 2 pi [? squared 3. ?]

PROFESSOR: Yeah. OK, that's right. So actually, what that does is to say the sixfold axis sits right up here. So I don't get any new sixfold axis rotation of 60 degrees here, followed by translation is the same as 60 degrees about here. So this is P6, and there is a lot of pure rotation axes combined with lattices. We've got P1, P2, P3, P4, and P6.

Now, what I will do eventually, when we're all done here, the plane groups are not derived in any book or set of tables that I am aware of. And next time, you will get some notes that do this in very slow motion fashion and give you all the individual steps. But the international tables does give you diagrams of the resulting plane groups, very nice carefully done figures along with the representative arrangement of motifs that they generate.

But we're not done yet. We have not let mirror planes enter the picture. And so unless there is dissension or debate, I'd like to consider what happens when we take a mirror plane, and we can combine that with two different kinds of lattices, a primitive rectangular net or a centered rectangular net, which is called C. And in order to do that, we need yet another combination theorem.

Here are the lattice points, and let me first derive the plane group that is called PM. And what I'll do is to put the-- let me use a squiggly line here, not because I'm excited or nervous about this, but just to distinguish it from the edges of the cell. So here is the operation sigma. The pattern that is going to be displayed by the plane group is once again just the pattern produced by a mirror plane hung at every lattice point.

But now we need a theorem. What we have here is the operation of reflection, followed by a translation that is perpendicular to the locus of the reflection line. And we will ask what is that? Again, you get the answer by just looking at once and for all, and say, if here is a first one and it is right-handed, and I reflect it to one number to a second one, number two, which is left-handed, and then move that by translation here to get number three, which stays left-handed if I move by translation. And ask now how was one related to three.

The chirality is changed. Reflection is the only thing available to us, and lo and behold, if I say there is a new mirror plane here, that tells me how this is related to this, and this one is related to this one. And this to this and this to this, so the answer to this question is that a reflection combined with a perpendicular translation is a new reflection operation sigma prime that is located at a distance removed from the first by 1/2 of that perpendicular translation.

And again, it's a plane old mirror plane just like the first one, but notice that the disposition of objects relative to the mirror plane in the center of the cell is quite distinct from the disposition of objects relative to the first mirror plane, so this is a second mirror plane. It is an independent mirror plane from the first. So that is plane group PM.

If there's symmetry in this business, you might ask is there a plane group AM? The answer is yes in three dimensions. There is a space group, PM, and there's a space group AM. So there's AM and PM, and there is symmetry, and all is well in the universe.

OK, we're making great progress here. And we'll be fairly well along before we have to bring things to a close. Let's do the second addition that's possible with a mirror plane. And that is to take a reflection operation and add it to the translations that are present in a centered rectangular net.

We've already done all the work for PM, so we can use that as a starting point. The pattern of this plane group is going to look like a pair of objects related by reflection. But now, we'll have an extra pair hung at the centered lattice point as well.

And the first thing we can note is that this mirror plane is no longer independent of the first one. What goes on at this mirror plane is something that's related to what happens at the origin lattice point and mirror plane, and therefore, these two mirror planes are going to do the same thing.

The pattern of this plane group, we've taken a mirror plane and added it to a centered rectangular net. This is called a C lattice, standing for centered. And correspondingly, the symbol for this plane group is CM.

We know how this one is related to this one. This one related to this one. And now, we've got something of a problem. All of these motifs are equivalent. How is this one related to this one? Or in more general terms, what we're asking is suppose I have a reflection operation sigma, and I add the translation not at right angles to it, as I did here, but place the translation at an angle with respect to the reflection plane.

So what we're going to do is to take a first motif that's right-handed, reflect it to a second one, which is left-handed. And then slide it along so that it sits up in the same position relative to this centered lattice point. So here's number three, translation leaves it left-handed. So I've taken the operation of reflection combined it with a translation that has a perpendicular component plus a parallel component, perpendicular and parallel meaning the orientation of these two components of the translation relative to the reflection operation.

Anybody want to hazard a guess on how that first one is related to the third one? Number one is right-handed. Number two is left-handed, so it's got to be reflection, right? If I put a reflection plane in here, this one ought to be tilted like this. That's not going to do the job. Anybody got any idea? Yeah.

AUDIENCE: What is that, T, T1 [INAUDIBLE]?

PROFESSOR: OK. T parallel plus T perpendicular means it's a component of this translation. This is T. This has a part T perpendicular, and it has a part T parallel relative to the initial mirror plane.

My friends, we have just stumbled headlong over a new type of symmetry operation, which we have discovered upon making this combination of mirror plane with a centered lattice. And it's come up to smack us rudely in the face even though we may not have been clever enough to think of it in advance.

This is a new type of operation, and it is an operation that cannot be reduced to one of the simple operations that we've defined so far. You've got to take two steps to get from number one to number three. The way you can do it is to reflect along a locus that is one half of the way along the perpendicular part of the translation, exactly the same location as we found the symmetry plane positioned when the translation was normal to the first mirror plane.

But yet we can't put the object down yet in the position that would be produced by translation, because our translation is inclined to the mirror plane. So I've got to take a second step. Reflect. Don't yet put it down yet. Before you put it down, slide it up parallel to the mirror plane by an amount that's equal to the part of the translation that is parallel to the mirror plane.

So to summarize this before all these words get too confusing, I'm saying that a reflection operation combined with a general translation that has a perpendicular component in the parallel component relative to the locus of reflection is going to be a new operation, which I'll write as sigma tau, a reflection part and a translation tau that is parallel to the mirror plane, and tau is equal to the part of the translation that is parallel to the mirror plane. Astounding.

This is a two-step operation that cannot be described in terms simpler than saying do two steps to do it. And we'll see as we go along, particularly into a three-dimensional space, that there are other two-step operations as well. Now you're all familiar with a pattern like this, because in very short order when New England's winter descends upon us, as you go slogging along from your room into the Institute, your footprints will make a pattern like that in the snow.

Exactly what we've got here. Reflect across and slide. Reflect across and slide. Reflect across and slide, and this is the glide component tau. And this is an operation that is called a glide plane.

And it's a new type of symmetry operation. It can only exist in a pattern, which has translational periodicity. And if we were not clever enough to invent it, we would see it as soon as we combined a mirror plane with a lattice that was non-primitive and had a translation parallel to it.

So it's a very, very descriptive name, the glide plane. Reflect and glide, reflect and glide, reflect and glide. It sounds like something you'd be doing in the Arthur Murray dance studio, very melodious. Reflect and glide, reflect and glide. It's a nice operation.

All right, so what has happened then when we add a mirror plane to a centered rectangular net is that we get a new operation coming in, something completely new. We've got a symbol to represent an individual operation, sigma the symbol for reflection with a subscript tau. And the pattern we've already drawn, but let's do it again in a tidy fashion.

Exactly as advertised, the pair of objects related by reflection hung at every lattice point of-- holy mackerel, look at this. That would give you the willies. That's the nature of the motif. Get that out of there. Mirror planes going through the lattice points.

Glide planes halfway in between. The mirror lines and the glide planes tell us how things on the right side, for example, of the lattice point are related to the motif of opposite chirality on the left-hand side of the centered lattice point. And this is a plane group that is called CM.

AUDIENCE: Question.

PROFESSOR: Yes, sir.

AUDIENCE: How can we just define a new operation that's a two-step. Seems like we could do this forever, define two steps of any new operation, like you said, with two steps.

PROFESSOR: That's a good question. We found this, because we tripped headlong over it. And we say OK, there it is. We've got to deal with it.

But you're right. How do you know that rotating once reflecting, and then turning end over end three times is not a new operation that cannot be decomposed, is the word that's used for it, into something simpler? The answer is you've got to try it.

If they're there, when we make these combinations of a symmetry operation, and now the symmetry operation can be a two-step symmetry operation, now that we've discovered that, combine that with lattice and with rotation and with reflection, and ask what is the relation between the motif at the beginning and the motif at the end. If you can't describe it any more simply other than a hop, skip and jump, you've got to introduce the hop, skip, and jump as an element that goes into the derivation of these groups.

Now before you get concerned and fill out an add/drop card, I have to reassure you that there are a couple of two-step operations that we have yet to discover, but there are no three-step operations that are necessary. Whew. Feel better now, don't you?

AUDIENCE: I was just going to say it's kind of arbitrary, because you have a and b, you're just applying the third [INAUDIBLE] T sub a and b, [INAUDIBLE] trivial multiplication tables.

PROFESSOR: Well, actually, this is something that is distinct. I mean here is the group, and you cannot describe the relation between everything that is in this pattern, which was obtained simply by taking the operation of reflection-- we know how to peacefully coexist with that-- and placing that pair of objects at lattice point of a centered rectangular net.

So that's nothing really freaky. I mean it's a straightforward addition, but if it's a group, you have to know when you combine all the operations pairwise that it'd be able to show that these operations are members of a group. And the answer in terms of the language of group theory, if you combine a reflection with a translation that has a component that is parallel to the reflection plane, then there is a new operation that comes up that has to be in the group multiplication table, and the operation has a reflection part and a translation part.

AUDIENCE: I have a question.

PROFESSOR: Yes.

AUDIENCE: How do we go from the lower left to the upper right in one operation?

PROFESSOR: The lower left to the upper right.

AUDIENCE: No, the upper right diagonally. Across the diagonal, the upper right corner of the square.

PROFESSOR: Upper right here?

AUDIENCE: Yeah. And the left-handed guide below that.

PROFESSOR: The left-handed guide below.

AUDIENCE: No. Down there.

PROFESSOR: Where is down there?

AUDIENCE: The lowest edge of the--

PROFESSOR: Down here?

AUDIENCE: Yeah.

PROFESSOR: OK. From this one to this one?

AUDIENCE: Yeah.

PROFESSOR: OK. What I can say is-- may sound like I'm slipping off the hook too easily-- we said that we really only want to consider operations that terminate within the cell, and the way I get from here to here is to reflect and then translate up by the diagonal translation. So that's something that lies outside the cell.

So I can always knock off an integral number of translations or add on an integral number of translations to any mapping transformation, modulo T. OK?

AUDIENCE: But surely, if you've done, one operation then it's [INAUDIBLE]?

PROFESSOR: From here to here? No, not necessarily. If I give you a very simple pattern and plane group P1, and you ask how do I get from this one here to this one that sits up here? If it's outside the cell, I've got to go translation that's outside the cell. But it's not any new translation or any new sort of operation. This is sort of in the same category. Yeah.

AUDIENCE: You can make a glide plane in the center of the cell.

PROFESSOR: Yeah, OK. Thank you. There is a glide operation that goes from here to here and then translates up by one full translation. But a glide operation that is an integral number of translations says that you're dealing with-- there's another object that is removed by a translation that is the same thing. So it is first and that simpler one that would be inside of the cell and would be only the unique sort of translation you need to consider.

Thank you. That's a good question. That was a good answer to his question. In fact, anybody want to take over? I didn't do well on that one. Yeah.

AUDIENCE: I was curious. Does the glide plane exist maybe in this case because we're not using a primitive cell? If you were to consider this [INAUDIBLE] as hexagonal?

PROFESSOR: OK, let me answer that question by saying that that's our first encounter with it. But now, that it exists, that is a symmetry element, which we should consider adding to lattices in addition to pure reflection. So let me proceed now to do another plane group since we had discovered the operation of glide.

And I will take a primitive rectangular lattice, and I will now add to the lattice point, not a mirror plane, but a glide plane. OK. The pattern is going to look like the pattern of glide, have things left and right on either side of the glide plane. The same thing here.

And there is the pattern. This is a pattern that's called PG. And what I have to ask is what is the guide operation sigma tau combined with, let's say this translation T1. And the answer is that if I reproduce number one to number two of opposite chirality by the glide operation sigma tau and then follow that by T1 the first and the third will be related by a new glide operation sigma tau prime that's located at half perpendicular part of the translation away from the first. And that really is a generalization of this relation here.

If we make the pure reflection operation a glide operation, combine it with a translation that is perpendicular to it, it turns out that the net result is a new glide operation, sigma tau prime. The two taus are equal, and this occurs at one half of the translation T perpendicular removed from the first.

And the complete generalization would be to say if I have a translation that has a parallel part and a perpendicular part relative to the glide plane, what I will get is a new glide operation sigma prime that's located at 1/2 of the perpendicular part of the translation from the first, and it will pick up a glide component that's equal to the original tau plus the part of the translation that's parallel to the glide plane. So this now is this theorem involving translation and reflection type operations in its most general form.

OK. So this is a new group called CM, and it consists of pairs of objects. Sorry, we did that earlier. This is PG, pairs of objects related by a glide plane, hung at every lattice point of the primitive rectangular lattice. Is there a CG?

Actually, there is not, and let me show you why, and then I think we're just about quitting time. If there is a centered lattice, and I hang a glide plane at the lattice points, we'll have glide planes in the locations of PG, the pattern will look like objects repeated by glide. At this lattice point, and now we're going to have another object hung at the centered lattice point, and it's going to be in positions like this.

Now, you can see just by looking at the pattern that there is going to be-- whoops, what did I do here? I want to move this one up to here, and I want to move this one up to here. This one should be in this location. If I look at that pattern, what I've done is to create halfway along the-- quarter of the way along the translation a pure reflection operation. And I can find that using my general theorem.

I have a glide operation, sigma tau. I combine with that the centered translation, which is 1/2 of T1 plus 1/2 of T2. I've taken 1/2 of T1 plus 1/2 of T2, that's the centering translation, combined that with sigma tau. I'll deftly jump to the side, so the people against the wall can see it.

I've taken a glide operation, combined it with this translation. This should be a new reflection type of operation that will be located at 1/2 of the perpendicular part of the translation, and that's at 1/2 of 1/2 of T1. And it will have a glide component tau prime, which is the original tau, which was 1/2 of T2.

And to that we add the parallel part of the translation, and that is 1/2 of T2. So this is rewritten in slightly different form, is simply a new glide plane sigma prime, which has a glide component equal to the entire translation T2 at 1/4 quarter of T1. And this is the same as a mirror plane at 1/4 T1.

And this, if we compare it with CM, is exactly the same thing. Identical to CM with an origin shift of 1/4 T1. So there is no CG in what we picked up in terms of rectangular nets and symmetry planes is PM, PG, and CN. So there are three groups involving orthogonal nets in a single symmetry plane.

That is a good place to wrap things up. And we'll next turn very, very quickly equipped with a set of notes, which summarizes the result, to what happens when we take two MM and put it into a rectangular net, and take two MM and put it into a centered rectangular net. And then things get interesting, because we've got two planes. We can make a both mirror planes. We can make them both glides, or make one a mirror plane and one a glide plane. So there are three possibilities with the addition of two MM to the net.

More than enough for one day. It's Thursday. Take the rest of the week and weekend off by doing no symmetry. There's no assignment, and we'll have at it again next Tuesday.