Solutions to Problem Set 2

Part I

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Solution (a) Let $u$ equal $8 - x^2$, and let $v = u^5$. Then $y$ equals,

$$y = \frac{x}{v}.$$  

By the quotient rule,

$$y' = \frac{1}{v^2}((x)v' - x(v')) = \frac{1}{v^2}(v - x(v')).$$  

By the chain rule,

$$v' = \frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx} = (5u^4)(-2x).$$  

When $x$ equals 3, $u$ equals $8 -(3)^2 = -1$ and $v$ equals $(-1)^5 = -1$. Thus $v'(3)$ equals $(5(-1)^4)(-2 \cdot 3) = -30$. Thus, $y'(3)$ equals,

$$y'(3) = \frac{1}{(-1)^2}((-1) - (3)(-30)) = -1 + 90 = 89.$$  

Therefore, the slope of the tangent line at $(3, -3)$ is,

$$y = 89(x - 3) + (-3), \quad y = 89x - 270.$$  

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Solution (b) Implicit differentiation gives,

\[ \frac{d}{dx} \left( \frac{1 - y}{1 + y} \right) = \frac{d(x)}{dx} = 1. \]

By the chain rule,

\[ \frac{d}{dx} \left( \frac{1 - y}{1 + y} \right) = \frac{\frac{1 - y}{1 + y}}{dx} \cdot \frac{dy}{dx}. \]

By the quotient rule,

\[ \frac{d}{dy} \left( \frac{1 - y}{1 + y} \right) = \frac{1}{(1 + y)^2}((-1)(1+y) - (1-y)(1+y)) = \frac{1}{(1 + y)^2}((-1)(1+y) - (1-y)(1)) = \frac{-2}{(1 + y)^2}. \]

Thus, implicit differentiation gives,

\[ \frac{-2}{(1 + y)^2} \frac{dy}{dx} = 1, \]

or,

\[ \frac{dy}{dx} = \frac{-2}{(1 + y)^2}. \]

To solve for \( x \), multiply both sides of the equation by \( 1 + y \) to get,

\[ 1 - y = x(1 + y) = x + xy. \]

Add \( y - x \) to each side of the equation to get,

\[ 1 - x = xy + y = (x + 1)y. \]

Divide each side of the equation to get,

\[ y = \frac{1 - x}{1 + x}. \]

By the quotient rule,

\[ y' = \frac{1}{(1 + x)^2}((-1)(1+y)'(1+x) - (1-x)(1+x)) = \frac{1}{(1 + x)^2}((-1)(1+x) - (1-x)(1)) = \frac{-2}{(1 + x)^2}. \]

Since \( y = (1 - x)/(1 + x) \), \( 1 + y \) equals,

\[ 1 + y = \frac{1 + x}{1 + x} + \frac{1 - x}{1 + x} = \frac{(1 + x) + (1 - x)}{1 + x} = \frac{2}{1 + x}. \]

Thus,

\[ \frac{-1}{2} (1 + y)^2 = \frac{-1}{2} \left( \frac{2}{1 + x} \right)^2 = \frac{-1}{2} \frac{4}{(1 + x)^2}. \]
Therefore,
\[ \frac{-1}{2} (1 + y)^2 = \frac{-2}{(1 + x)^2}. \]

So the two answers for \( y' \) are equivalent.

**Solution (c)**
The fraction simplifies,
\[ y = \frac{x}{1 + x} = \frac{(1 + x) - 1}{1 + x} = \frac{1 + x}{1 + x} - \frac{1}{1 + x} = 1 - \frac{1}{1 + x}. \]

Let \( z \) equal \( 1/(1 + x) \). Then \( y \) equals \( 1 - z \). So \( y' \) equals \( -z' \). So \( y'' \) equals \( -z'' \). Clearly, for every positive integer \( n \), \( y^{(n)} \) equals \( -z^{(n)} \). By the same argument as in Example 2 on p. 109,
\[ z^{(n)} = (-1)^n n!(x + 1)^{-(n+1)}. \]

Thus, for every positive integer \( n \),
\[ y^{(n)} = (-1)^{n+1} n!(x + 1)^{-(n+1)}. \]

**Solution (d)**
The equation,
\[ y = \log_a (x + \sqrt{x^2 - 1}), \]
is equivalent to the equation,
\[ a^y = x + \sqrt{x^2 - 1}. \]

Subtract \( x \) from each side to get,
\[ a^y - x = \sqrt{x^2 - 1}, \]
and then square each side to get,
\[ (a^y - x)^2 = x^2 - 1. \]

Expanding the left-hand-side gives,
\[ (a^y)^2 - 2xa^y + x^2 = x^2 - 1. \]

Cancelling \( x^2 \) from each side gives,
\[ (a^y)^2 - 2xa^y = -1. \]

Adding \( 2xa^y + 1 \) to each side of the equation gives,
\[ (a^y)^2 + 1 = 2xa^y. \]

The expression \( 2a^y \) is always nonzero. Thus it is valid to divide each side by \( 2a^y \), giving,
\[ x = [(a^y)^2 + 1]/(2a^y) = [a^y + 1/a^y]/2. \]
Of course $1/a^y$ equals $a^{-y}$. So this simplifies to,

$$x = \frac{(a^y + a^{-y})}{2}.$$

**Solution (e)** Because $1/2$ is less than 1, $\log(1/2)$ is less than $\log(1) = 0$. Thus $\log(1/2)$ is negative. For every pair of real numbers $a < b$ and every negative number $c$, $ac$ is greater than $bc$, not less than $bc$. Therefore, the correct inequality is,

$$1 \cdot \log\left(\frac{1}{2}\right) > 2 \cdot \log\left(\frac{1}{2}\right).$$

The remainder of the argument is correct, and eventually leads to the true inequality,

$$\frac{1}{2} > \frac{1}{4}.$$

**Solution (f)** By the product rule,

$$\frac{d}{dx}(x^2 e^{-x^2}) = \frac{d}{dx}(x^2 e^{-x^2}) + x^2 \frac{d}{dx}(e^{-x^2}) = 2x e^{-x^2} + x^2 \frac{d}{dx}(e^{-x^2}).$$

Let $u$ equal $-x^2$ and let $v$ equal $e^u$. Thus $v$ equals $e^{-x^2}$. By the chain rule,

$$\frac{dv}{dx} = \frac{dv}{du} \frac{du}{dx}.$$ 

Also,

$$\frac{d(e^u)}{du} = e^u, \text{ and } \frac{d(-x^2)}{dx} = -2x.$$ 

Plugging in,

$$\frac{dv}{dx} = e^u(-2x) = e^{-x^2}(-2x) = -2x e^{-x^2}.$$ 

Thus,

$$\frac{d}{dx}(x^2 e^{-x^2}) = 2x e^{-x^2} + x^2(-2x e^{-x^2}) = 2x e^{-x^2} - 2x^3 e^{-x^2} = -2x(x^2 - 1)e^{-x^2}.$$ 

**Solution (g)** Let $u$ equal $\ln(y)$. Then, using rules of logarithms,

$$u = \ln(y) = \ln\left(\frac{x^2 + 3}{x + 5}\right)^{1/5} = \frac{1}{5} \ln\left(\frac{x^2 + 3}{x + 5}\right) = \frac{1}{5} \ln(x^2 + 3) - \frac{1}{5} \ln(x + 5).$$ 

Thus,

$$\frac{du}{dx} = \frac{1}{5} \frac{d}{dx}(\ln(x^2 + 3)) - \frac{1}{5} \frac{d}{dx}(\ln(x + 5)).$$
Let \( v \) equal \( x^2 + 3 \). By the chain rule,
\[
\frac{d \ln(v)}{dx} = \frac{d \ln(v)}{dv} \frac{dv}{dx} = \frac{1}{v} (2x) = \frac{2x}{x^2 + 3}.
\]
Let \( w \) equal \( x + 5 \). By the chain rule,
\[
\frac{d \ln(w)}{dx} = \frac{d \ln(w)}{dw} \frac{dw}{dx} = \frac{1}{w} (1) = \frac{1}{x + 5}.
\]
Thus,
\[
\frac{du}{dx} = \frac{2x}{5(x^2 + 3)} - \frac{1}{5(x + 5)}.
\]
On the other hand,
\[
\frac{du}{dx} = \frac{d \ln(y)}{dx} = \frac{1}{y} \frac{dy}{dx}.
\]
Therefore,
\[
\frac{dy}{dx} = y \frac{du}{dx} = \sqrt{\frac{x^2 + 3}{x + 5}} \left( \frac{2x}{5(x^2 + 3)} - \frac{1}{5(x + 5)} \right).
\]

**Solution (h)** By the double-angle formula,
\[
\cos(2\phi) = (\cos(\phi))^2 - (\sin(\phi))^2 = 2(\cos(\phi))^2 - [(\cos(\phi))^2 + (\sin(\phi))^2] = 2(\cos(\phi))^2 - 1.
\]
Substituting \( 2\theta \) for \( \phi \) gives,
\[
\cos(4\theta) = 2(\cos(2\theta))^2 - 1.
\]
Substituting \( 2\theta \) in \( \cos(2\theta) = 2(\cos(\theta))^2 - 1 \) gives,
\[
\cos(4\theta) = 2[2(\cos(\theta))^2 - 1]^2 - 1 = 2[4(\cos(\theta))^4 - 4(\cos(\theta))^2 + 1] - 1
= 8(\cos(\theta))^4 - 8(\cos(\theta))^2 + 1.
\]

**Solution (i)** First of all, \( \ln(x^2) \) equals \( 2 \ln(x) \). Thus,
\[
y = \sin(2 \ln(x)).
\]
Let \( u \) equal \( 2 \ln(x) \). Thus \( y \) equals \( \sin(u) \). By the chain rule,
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.
\]
Of course,
\[
\frac{du}{dx} = \frac{d(2 \ln(x))}{dx} = \frac{2}{x}.
\]
Also,
\[ \frac{d\sin(u)}{du} = \cos(u). \]
Thus,
\[ \frac{dy}{dx} = \cos(u) \frac{2}{x} = \frac{2\cos(2\ln(x))/x.}{x} \]

**Solution (j)** Writing the functions out in terms of \(\sin(x)\) and \(\cos(x)\),
\[ y = \left( \frac{\cos(x)}{\sin(x)} + \frac{1}{\sin(x)} \right)^2 = \left( \frac{\cos(x) + 1}{\sin(x)} \right)^2. \]
Let \(u = \frac{(\cos(x) + 1)/\sin(x)}{.}\) Then \(y = u^2\). By the chain rule,
\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 2u \frac{du}{dx}. \]
Let \(v = \cos(x) + 1\) and let \(w = \sin(x)\). Then \(u = v/w\). By the quotient rule,
\[ \frac{du}{dx} = \frac{1}{w^2} \left( \frac{dv}{dx} \frac{w}{dx} - v \frac{dw}{dx} \right). \]
Also, \(dv/dx\) equals \(-\sin(x)\) and \(dw/dx\) equals \(\cos(x)\). Thus,
\[ \frac{du}{dx} = \frac{1}{(\sin(x))^2} \left( (-\sin(x)) \sin(x) - (\cos(x) + 1) \cos(x) \right) \]
\[ = \frac{1}{(\sin(x))^2} \left( -\sin(x)^2 - (\cos(x))^2 - \cos(x) \right) = \frac{-(\cos(x) + 1)}{(\sin(x))^2}. \]
Therefore,
\[ \frac{dy}{dx} = \frac{2(\cos(x) + 1) - (\cos(x) + 1)}{\sin(x)} \frac{(\sin(x))^2}{(\sin(x))^2} = \]
\[ -2(\cos(x) + 1)^2/(\sin(x))^3. \]

**Part II** (30 points)

**Problem 1** (5 points) Find the equation of the tangent line to the graph of \(y = e^{571x}\) containing the point \((102\pi, 0)\). (This is not a point on the graph; it is a point on the tangent line.)

**Solution to Problem 1** Denote 571 by the symbol \(n\). Denote \(102\pi\) by the symbol \(a\). The derivative of \(e^{nx}\) equals \(n e^{nx}\). Thus the slope of the tangent line to \(y = e^{nx}\) at the point \((b, e^{nb})\) equals \(n e^{nb}\). So the equation of the tangent line to \(y = e^{nx}\) at \((b, e^{nb})\) is,
\[ y = ne^{nb}(x - b) + e^{nb} = ne^{nb}x - (nb - 1)e^{nb}. \]
If \((a, 0)\) is contained in this line, then the equation holds for \(x = a\) and \(y = 0\),
\[ 0 = ne^{nb}a - (nb - 1)e^{nb} = (na - nb + 1)e^{nb}. \]
Since $e^{ab}$ is not zero, dividing by $e^{nb}$ gives,

$$na - nb + 1 = 0.$$ 

This can be solved to determine the one unknown in the equation, $b$:

$$b = (na + 1)/n.$$ 

Substituting this in gives the equation of the tangent line to $y = e^{nx}$ containing $(a, 0)$,

$$y = ne^{(na+1)x} - nae^{(na+1)}.$$

**Problem 2** (5 points)

(a) (2 points) What does the chain rule say if $y = x^a$ and $u = y^b$? The constants $a$ and $b$ are fractions.

**Solution to (a)** First of all, $u$ equals $y^b$, which equals $(x^a)^b$. By the rules for exponents, this equals $x^{ab}$. According to the chain rule,

$$\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = (by^{b-1})(ax^{a-1}).$$

Substituting in $y = x^a$ gives,

$$\frac{du}{dx} = (b(x^a)^{b-1})(ax^{a-1}).$$

Using the rules for exponents, this equals,

$$\frac{du}{dx} = (bx^{a(b-1)})(ax^{a-1}) = abx^{ab-a}x^{a-1} = abx^{ab-1}.$$ 

This is precisely what the chain rule should give, since, setting $c = ab$,

$$\frac{d(x^c)}{dx} = cx^{c-1} = abx^{ab-1}.$$ 

(b) (3 points) Using the chain rule, give a very short explanation of the formula from Problem 3, Part II of Problem Set 1.

**Solution to (b)** Let $u$ equal $ax$ and let $y$ equal $f(u)$. Then $y$ equals $f(ax)$, which is $g(x)$. Thus $g'(x)$ equals $dy/dx$. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = f'(u)(ax)' = f'(ax)(a)$$

$$= af'(ax).$$
Problem 3 (10 points) A bank offers savings accounts and loans. For an initial deposit of $A$ dollars in a savings account with continuously compounded interest at an annual rate $a$, after $t$ years the bank owes the customer $A(1+a)^t$ dollars (neglecting fees). For an initial loan of $B$ dollars with continuously compounded interest at an annual rate $b$, after $t$ years the customer owes the bank $B(1+b)^t$ dollars (neglecting fees). To make a profit, the bank sets rate $b$, the interest rate for loans, higher than rate $a$, the interest rate for savings. To simplify computations, introduce $\alpha = \ln(1+a)$ and $\beta = \ln(1+b)$.

Customer 1 deposits $A$ dollars in a savings account. The bank immediately loans a smaller amount of $B$ dollars to Customer 2. After $t$ years, the bank’s net gain from the two transactions together is,

$$G(t) = Be^{\beta t} - Ae^{\alpha t}. \quad (1)$$

In the long run, which is to say, when $t$ is very large, $G(t)$ is positive and the bank has made a gain. However, for $t$ small, $G(t)$ is negative and the bank has a net liability,

$$L(t) = -G(t) = Ae^{\alpha t} - Be^{\beta t}. \quad (2)$$

The liability for the savings account alone is,

$$M(t) = Ae^{\alpha t}. \quad (3)$$

In these equations, $A, B, \alpha$ and $\beta$ are positive constants, and $t$ is the independent variable.

(a) (5 points) Find the moment $t = T$ when the derivative $L'(T)$ equals 0. Assume that $\alpha A$ is greater than $\beta B$. Also, leave your answer in the form,

$$e^{(\beta-\alpha)T} = \text{something.}$$

Remark. After Lecture 10, we will learn that $T$ is the moment when $L(t)$ has its largest value. In other words, if at time $t$ Customer 1 withdraws all money, and Customer 2 repays all money, the bank loses the maximum amount when $t = T$.

Solution to (a) Using the chain rule,

$$L'(t) = A(e^{\alpha t})' - B(e^{\beta t})' = A(\alpha e^{\alpha t}) - B(\beta e^{\beta t}) = \alpha Ae^{\alpha t} - \beta Be^{\beta t}.$$

By definition of $T$, $L'(T)$ equals 0. Thus,

$$\beta Be^{\beta T} = \alpha Ae^{\alpha T}.$$

Dividing each side of the equation by $\beta Be^{\alpha T}$ gives,

$$e^{\beta T}/e^{\alpha T} = \frac{\alpha A}{\beta B}.$$

Using rules of exponents, this is,

$$e^{(\beta-\alpha)T} = \frac{(\alpha A)/(\beta B)}.\]
It is worth remarking that to make $T$ small, the bank may maximize the fraction $B/A$ of money loaned to money deposited and the bank may maximize the fraction $\beta/\alpha$ (although if $\beta$ is too high or $\alpha$ too low, customers are discouraged from using the bank).

(b)(5 points) Consider the ratio $L(t)/M(t)$. Using your answer to (a), determine $L(T)/M(T)$. Simplify your answer as much as possible. How does this ratio depend on the amounts $A$ and $B$?

**Solution to (b)** The ratio $L(t)/M(t)$ equals,

$$\frac{L(t)}{M(t)} = \frac{A e^{\alpha t} - B e^{\beta t}}{A e^{\alpha t}} = 1 - \frac{B}{A} e^{(\beta - \alpha)t}.$$

By the formula from the **Solution to (a)** $e^{(\beta - \alpha)T}$ equals $(\alpha A)/(\beta B)$. Plugging this in,

$$\frac{L(T)}{M(T)} = 1 - \frac{B}{A} \frac{\alpha A}{\beta B} = 1 - \frac{\alpha}{\beta}.$$

In particular, this is independent of $A$ and $B$.

**Remark.** From the formulas, the bank’s strategy is clear. First, adjust the ratio $\beta/\alpha$ to the highest level allowed by law and compatible with the market’s demands. Then, for fixed $\alpha$ and $\beta$, the ratio $L(T)/M(T)$ is independent of $A$ and $B$. So the maximal liability $L(T)$ is proportional to $M(T)$. Since $M(T)$ is an increasing function, the strategy is to minimize $T$, by maximizing the ratio $B/A$. (Of course this ratio will always be less than 1, since some fraction of all capital goes to the federal reserve, some fraction is used to cover operating expenses, etc.)

**Problem 4** (10 points) Let $A$, $\beta$, $\omega$ and $t_0$ be positive constants. Let $f(t)$ be the function,

$$f(t) = A e^{-\beta t} \cos(\omega(t - t_0)).$$

(a)(5 points) Compute $f'(t)$ and $f''(t)$. Simplify your answer as much as possible.

**Solution to (a)** Let $s$ equal $t - t_0$. Then $f(t) = g(s)$, where,

$$g(s) = B e^{-\beta s} \cos(\omega s),$$

and $B = A e^{-\beta t_0}$. The derivative $ds/dt$ equals 1. Thus, according to the chain rule,

$$\frac{df}{dt} = \frac{dg}{ds}.$$

Using the product rule and the chain rule, this equals,

$$\frac{dg}{ds} = B (e^{-\beta s})' \cos(\omega s) + B e^{-\beta s} (\cos(\omega s))' = B \omega B e^{-\beta s} \cos(\omega s) - B \omega e^{-\beta s} \sin(\omega s).$$
In particular, when $B$ equals 1, this gives,

$$\frac{d}{ds}(e^{-\beta s}\cos(\omega s)) = -\beta e^{-\beta s}\cos(\omega s) - \omega e^{-\beta s}\sin(\omega s)).$$

The second derivative $g''(s)$ involves the derivative of $e^{-\beta s}\cos(\omega s)$, but it is also involves the derivative of $e^{-\beta s}\sin(\omega s)$. Using the chain rule and the product rule,

$$\frac{d}{ds}(e^{-\beta s}\sin(\omega s)) = (e^{-\beta s})'\sin(\omega s) + e^{-\beta s}(\sin(\omega s))' = -\beta e^{-\beta s}\sin(\omega s) + \omega e^{-\beta s}\cos(\omega s).$$

As above,

$$\frac{d^2 f}{dt^2} = \frac{d^2 g}{ds^2}.$$  

This is,

$$\frac{d}{ds}\left(\frac{dg}{ds}\right) = -B\beta[e^{-\beta s}\cos(\omega s)]' - B\omega[e^{-\beta s}\sin(\omega s)]' = -B\beta[-\beta e^{-\beta s}\cos(\omega s) - \omega e^{-\beta s}\sin(\omega s)] - B\omega[-\beta e^{-\beta s}\sin(\omega s) + \omega e^{-\beta s}\cos(\omega s)] = (B\beta^2 - B\omega^2)e^{-\beta s}\cos(\omega s) + (B\beta\omega + B\beta\omega)e^{-\beta s}\sin(\omega s) = B(\beta^2 - \omega^2)e^{-\beta s}\cos(\omega s) + 2B\beta e^{-\beta s}\sin(\omega s).$$

Back-substituting $s = t - t_0$ and $Be^{-\beta s} = Ae^{-\beta t}$ gives,

$$f'(t) = -A\beta e^{-\beta t}\cos(\omega(t - t_0)) - A\omega e^{-\beta t}\sin(\omega(t - t_0)),$$

and,

$$f''(t) = A(\beta^2 - \omega^2)e^{-\beta t}\cos(\omega(t - t_0)) + 2A\beta e^{-\beta t}\sin(\omega(t - t_0)).$$

(b) (5 points) Using your answer to [a] find nonzero constants $c_0$, $c_1$ and $c_2$ for which the function

$$c_2 f''(t) + c_1 f'(t) + c_0 f(t),$$

always equals 0.

**Solution to (b)** From the Solution to (a), $f'(t) + \beta f(t)$ equals $-Ae^{-\beta t}(\omega \sin(\omega(t - t_0)))$. Plugging this into the formula for $f''(t)$ gives,

$$f''(t) = (\beta^2 - \omega^2)f(t) - 2\beta(f'(t) + \beta f(t)) = -2\beta f' - (\beta^2 + \omega^2)f(t).$$

Simplifying gives,

$$f''(t) + 2\beta f'(t) + (\omega^2 + \beta^2)f(t) = 0.$$  

In fact, every solution is of the form,

$$c_2 = c, c_1 = 2\beta c, c_0 = \omega^2 + \beta^2,$$

for some nonzero $c$. 

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