18.01 PRACTICE FINAL, FALL 2003

Problem 1 Find the following definite integral using integration by parts.

\[ \int_{0}^{\pi} x \sin(x) \, dx. \]

Problem 2 Find the following antiderivative using integration by parts.

\[ \int x \sin^{-1}(x) \, dx. \]

Problem 3 Use L'Hospital's rule to compute the following limits.

(a) \( \lim_{x \to 0} \frac{e^x - b^x}{x^2} \), \( 0 < a < b \).

(b) \( \lim_{x \to 1} \frac{4x^3 - 5x + 1}{\ln x} \).

Problem 4 Determine whether the following improper integral converges or diverges.

\[ \int_{1}^{\infty} e^{-x^2} \, dx. \]

(Hint: Compare with another function.)

Problem 5 You wish to design a trash can that consists of a base that is a disk of radius \( r \), cylindrical walls of height \( h \) and radius \( r \), and the top consists of a hemispherical dome of radius \( r \) (there is no disk between the top of the walls and the bottom of the dome; the dome rests on the top of the walls). The surface area of the can is a fixed constant \( A \). What ratio of \( h \) to \( r \) will give the maximum volume for the can? You may use the fact that the surface area of a hemisphere of radius \( r \) is \( 2\pi r^2 \), and the volume of a hemisphere is \( \frac{2}{3} \pi r^3 \).

Problem 6 A point on the unit circle in the \( xy \)-plane moves counterclockwise at a fixed rate of \( 1 \, \text{radian/second} \). At the moment when the angle of the point is \( \theta = \frac{\pi}{4} \), what is the rate of change of the distance from the particle to the \( y \)-axis?

Problem 7 Compute the following integral using a trigonometric substitution. Don’t forget to back-substitute.

\[ \int \frac{x^2}{\sqrt{1 - x^2}} \, dx. \]

Hint: Recall the half-angle formulas, \( \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)) \), \( \sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)) \).

Problem 8 Compute the volume of the solid of revolution obtained by rotating about the \( x \)-axis the region in the 1\(^{st} \) quadrant of the \( xy \)-plane bounded by the axes and the curve \( x^4 + r^2 y^2 = r^4 \).

Problem 9 Compute the area of the surface of revolution obtained by rotating about the \( y \)-axis the portion of the lemniscate \( r^2 = 2a^2 \cos(2\theta) \) in the 1\(^{st} \) quadrant, i.e., \( 0 \leq \theta \leq \frac{\pi}{4} \).

Problem 10 Compute the area of the \( \text{lune} \) that is the region in the 1\(^{st} \) and 3\(^{rd} \) quadrants contained inside the circle with polar equation \( r = 2a \cos(\theta) \) and outside the circle with polar equation \( r = a \).

Problem 11 Find the equation of every tangent line to the hyperbola \( C \) with equation \( y^2 - x^2 = 1 \), that contains the point \( (0, \frac{1}{2}) \).

Date: Fall 2003.
Problem 12 Compute each of the following integrals.

(a) \( \int \sec^3(\theta) \tan(\theta) d\theta \).
(b) \( \int \frac{x^2 - 1}{x(x+1)^2} dx \).
(c) \( \int \frac{2x-1}{2x^2 - 2x + 3} dx \).
(d) \( \int \sqrt{e^{3x}} dx \).