Lecture 1. September 8, 2005

**Homework.** Problem Set 1 Part I: (a)–(e); Part II: Problems 1 and 2.

**Practice Problems.** Course Reader: 1B-1, 1B-2

1. **Velocity.** Displacement is \( s(t) \). *Increment* from \( t_0 \) to \( t_0 + \Delta t \) is,
\[
\Delta s = s(t_0 + \Delta t) - s(t_0).
\]
*Average velocity* from \( t_0 \) to \( t_0 + \Delta t \) is,
\[
v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}.
\]

**Velocity**, or *instantaneous velocity*, at \( t_0 \) is,
\[
v(t_0) = \lim_{\Delta t \to 0} v_{\text{ave}} = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}.
\]
This is a *derivative*, \( v(t) \) equals \( s'(t) = ds/dt \). The derivative of velocity is **acceleration**, \( a(t_0) = v'(t_0) = \lim_{\Delta t \to 0} \frac{v(t_0 + \Delta t) - v(t_0)}{\Delta t} \).

**Example.** For \( s(t) = -5t^2 + 20t \), first computed velocity at \( t = 1 \) is,
\[
v(1) = \lim_{\Delta t \to 0} 10 - 5\Delta t = \boxed{10}.
\]
Then computed velocity at \( t = t_0 \) is,
\[
v(t_0) = \lim_{\Delta t \to 0} -10t_0 + 10 - 5\Delta t = \boxed{-10t_0 + 20}.
\]
Finally, computed acceleration at \( t = t_0 \) is,
\[
a(t_0) = \lim_{\Delta t \to 0} -10 = \boxed{-10}.
\]

2. **Derivative.** Let \( y = f(x) \) be a *dependent variable* depending on an *independent variable* \( x \), varying freely. The *increment* of \( y \) from \( x_0 \) to \( x_0 + \Delta x \) is,
\[
\Delta y = f(x_0 + \Delta x) - f(x_0).
\]
The difference quotient or average rate-of-change of $y$ from $x_0$ to $x_0 + \Delta x$ is,

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The derivative of $y$ (or $f(x)$) with respect to $x$ at $x_0$ is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

3. Examples in science and math.

(i) Economics. Marginal cost is the derivative of cost with respect to some other variable, for instance, the quantity purchased.

(ii) Thermodynamics. The ideal gas law relating pressure $p$, volume $V$, and temperature $T$ of a gas is,

$$pV = nRT.$$

Under isothermal conditions, $T$ is a constant $T_0$ so that,

$$p(V) = \frac{nRT_0}{V}.$$

Under adiabatic conditions (i.e., no transfer of heat), $pV^\gamma$ is a constant $K$. Using this to eliminate $p$ gives,

$$T(V) = \frac{K}{nR V^{\gamma-1}}.$$

As this illustrates, the independent variable, dependent variable and constants in an equation very much depend on the problem to be solved.

(iii) Biology. Exponential population growth models the population $N(t)$ after $t$ years as,

$$N(t) = N_0 e^{rt},$$

where $e^x$ is the exponential function, $N_0$ is initial population, and $r$ is a growth factor. Later we will see, $N'(t) = rN(t)$, i.e., the population grows at a rate proportional to the size of the population.

(iv) Geometry. The volume of a right circular cone is,

$$V = \frac{1}{3} A \times h.$$

where $A$ is the base area of the cone and $h$ is the height of the cone. The radius $r$ of the base is proportional to the height,

$$r(h) = ch,$$
for some constant $c$. Since $A = \pi r^2$, this gives,

$$V(h) = \frac{\pi}{3} c^2 h^3. $$

The derivative is,

$$\frac{dV}{dh} = \pi c^2 h^2 = \pi r^2 = A.$$

This is very reasonable. In some sense, this explains the classical formula for the volume of a cone.